Introduction to Hawking Radiation in The Tunneling Pictures

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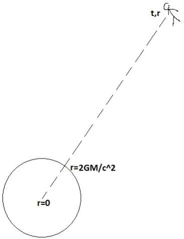
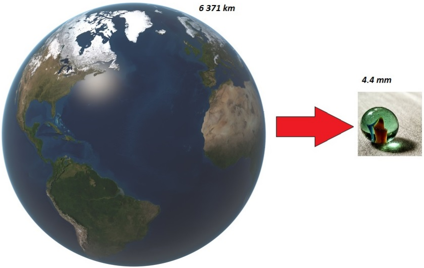
Outline

- Black hole = (true) singularity + event horizon
- From Newtonian gravity: $v_{esc} = \sqrt{2GMr^{-1}}$
- Intro: flat spacetime
  $ds^2 = c^2 d\tau^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 = -c^2 dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2 d\phi^2 \right)$
- The presence of energy (eg. mass) curves the spacetime, ex.
  $ds^2 = -f(r)\, c^2 dt^2 + g(r)^{-1}\, dr^2 + r^2 \left(d\theta^2 + \sin^2 d\phi^2 \right)$
- The simplest non-flat solution to the vacuum Einstein equation: Schwarzschild

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \frac{dr^2}{\left(1 - \frac{2GM}{c^2 r}\right)} + r^2 \left(d\theta^2 + \sin^2 d\phi^2 \right)$$



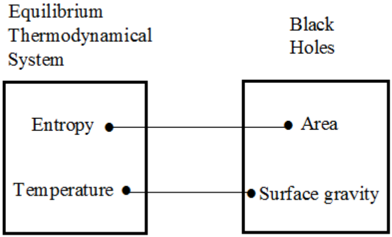


- What if black holes thermodynamically dead ?

- The second law of thermodynamics states that the total entropy can never decrease over time for an isolated system, i.e. $\delta S \geq 0$.
- Somehow, there should be an entropy associated to the black hole whose change follows the conservation of energy.

Black holes mechanics and thermodynamics

- The conservation of energy for black holes: $\delta M = (\kappa/8\pi)\delta A_H + \Omega_H \delta J$; Bardeen, Carter, Hawking, 1973.
- 1st law of thermodynamics: $\delta E = T\delta S - P\delta V$.
- Black hole mechanics and thermodynamics relations,



- Bekenstein-Hawking entropy: $S_{BH} = \frac{A_{BH}}{4}$.
- Black hole temperature: $T_H = \frac{\kappa}{2\pi}$.

Hawking original derivation of BH radiation

- Requires familiarity on QFT in curved background..

$$\Phi = \sum_j \left\{ f_j \hat{a}_j + \bar{f}_j \hat{a}_j^\dagger \right\} \to \sum_j \left\{ p_j \hat{b}_j + \bar{p}_j \hat{b}_j^\dagger + q_j \hat{c}_j + \bar{q}_j \hat{c}_j^\dagger \right\}$$

$$\nabla^2 \Phi = 0$$

Particle Creation by Black Holes

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Abstract. In the classical theory black holes can only absorb and not emit particles. However it is shown that quantum mechanical effects cause black holes to create and emit particles as if they were hot bodies with temperature $\frac{\hbar\kappa}{2\pi k} \approx 10^{-6} \left(\frac{M_\odot}{M}\right)$ °K where κ is the surface gravity of the black hole. ....

Commun. math. Phys. 43, 199—220 (1975)

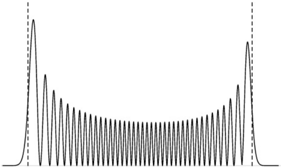
- Schrodinger eqtn.

$$\frac{\hbar^2}{2m}\nabla^2\Psi + (E-U)\Psi = 0$$

- One can employ the ansatz $\Psi \sim e^{iS(x)/\hbar}$, where the function $S(x) = S_0 + \left(\frac{\hbar}{i}\right) S_1 + \left(\frac{\hbar}{i}\right)^2 S_2 + \dots$
- If $|d\lambda/dx| \ll 1$, or possibly in case of large momentum case, we can have the solution from S_0 [2]

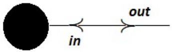
$$\Psi = \frac{C_+}{\sqrt{p}} \exp\left(\frac{i}{\hbar}\int pdx\right) + \frac{C_-}{\sqrt{p}} \exp\left(\frac{-i}{\hbar}\int pdx\right)$$

- For a time dependent Ψ, we have $S_0 = -Et \pm \int pdx$, i.e. the mechanical action of the particle; $-\frac{\partial S}{\partial t} = E \ , \ \frac{\partial S}{\partial x} = p$.
- In case of $E < U$, $\Psi = \frac{C_+}{\sqrt{|p|}} \exp\left(\frac{1}{\hbar}\int |p|\, dx\right) + \frac{C_-}{\sqrt{p}} \exp\left(\frac{-1}{\hbar}\int |p|\, dx\right)$, or by taking $\mathrm{Im}\int pdx$.

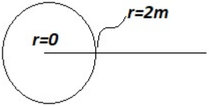


[2] $p = \sqrt{2m(E-U)}$, Landau and Lifshitz , Quantum Mechanics , Non-Relativistic Theory.

- Tunneling process is the common way to explain radiation.



- Radial: $d\theta = d\phi = 0$ and null: $ds^2 = 0$.



- Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right) d\tilde{t}^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

- We need a coordinate that is not singular at the horizon, for example Painleve

$$t = \tilde{t} + 2\sqrt{2mr} + 2m \ln\left(\frac{\sqrt{r} - \sqrt{2m}}{\sqrt{r} + \sqrt{2m}}\right)$$

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + 2\sqrt{\frac{2m}{r}} dtdr + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

- The coordinate singularity $r = 2m$ is removed, the true singularity $r = 0$ is still there, and the spacetime is stationary.

- The in(out) $[-(+)]$ null radial geodesic:

$$0 = -\left(1 - \frac{2m}{r}\right) dt^2 + 2\sqrt{\frac{2m}{r}} dtdr + dr^2 \quad \rightarrow \quad \frac{dr}{dt} = \pm 1 - \sqrt{\frac{2m}{r}}$$

- Near the horizon, where the tunneling process takes place, we can have

$$\dot{r} \approx \frac{(r-2m)}{4m}$$

- Imaginary part of the action for an outgoing particle from r_{in} to r_{out}

$$\mathrm{Im}S = \mathrm{Im} \int_{r_{in}}^{r_{out}} p dr = \mathrm{Im} \int_{r_{in}}^{r_{out}} \int_0^p dp' dr = \mathrm{Im} \int_{r_{in}}^{r_{out}} \int_0^E \frac{dH}{\dot{r}} dr$$

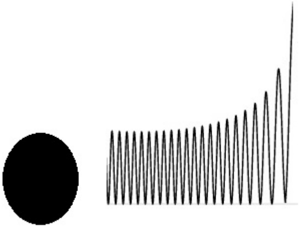
where the Hamilton's equation $\dot{r} = \left.\frac{dH}{dp}\right|_r$ has been employed[3].

- Using WKB approach: the particle has tunneling rate $e^{-2\hbar^{-1}\mathrm{Im}S}$ where $\mathrm{Im}S = 4\pi EM$.
- Expression $e^{-2\hbar^{-1}\mathrm{Im}S}$ takes the form of Boltzmann factor with energy E and (Hawking) temperature $T_H = 1/(8\pi M)$.

[3]Note that in doing integration over r, there is a pole at $r = 2m$.

A little bit on WKB and particle in a black hole background

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$



A particle losses its energy as it moves toward infinity from near a black hole.

- Second method: complex path.
- Allow the spacetime coordinate to be complex, i.e. $x = \mathrm{Re}x + i\mathrm{Im}x$, where x is four-coordinate, (t, r, θ, ϕ).
- We can work in the original Schwarzschild metric without Painleve transformation.
- Massless scalar particle in curved background

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi(x) = 0$$

- Use WKB ansatz $\Phi = e^{iS(x)/\hbar}$, where $S(x) = S_0 + \left(\frac{\hbar}{i}\right) S_1 + \left(\frac{\hbar}{i}\right)^2 S_2 + \ldots$
- Eqtn. for $S_)$:

$$\left(\frac{\partial S_0}{\partial t}\right)^2 = \left(1 - \frac{2m}{r}\right)^2 \left(\frac{\partial S_0}{\partial r}\right)^2 \rightarrow \frac{\partial S_0}{\partial t} = \pm\left(1 - \frac{2m}{r}\right)\frac{\partial S_0}{\partial r} \qquad (4.1)$$

where $+(-)$ associate to in(out)going particles.

- Lets write the action: $S_0 = Et + \tilde{S}_0(r)$, so eq (4.1) can be read as

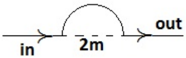
$$\frac{\partial S}{\partial r} = \pm\frac{Er}{r - 2m} \rightarrow \tilde{S}_0 = \pm\int_{r_{in}}^{r_{out}} \frac{Er\,dr}{r - 2m}$$

- Wave solutions:

$$\Phi_{in} = \exp\left(-\frac{i}{\hbar}\left(Et + E\int_{r_{in}}^{r_{out}} \frac{rdr}{r-2m}\right)\right) \;,\; \Phi_{out} = \exp\left(-\frac{i}{\hbar}\left(Et - E\int_{r_{in}}^{r_{out}} \frac{rdr}{r-2m}\right)\right)$$

- Condition from classical limit, i.e. $\hbar \to 0$, $|\Phi_{in}|^2 = 1$, yields $\mathrm{Im}t = -\mathrm{Im}\int_{r_{in}}^{r_{out}} \frac{rdr}{r-2m}$.
- Consequently,

$$P_{out} = |\Phi_{out}|^2 = \exp\left(-\frac{4E}{\hbar}\mathrm{Im}\int_{r_{in}}^{r_{out}} \frac{rdr}{r-2m}\right) = \exp\left(-\frac{8\pi ME}{\hbar}\right)$$



- Using the principle[4] of "detailed balance" [Hartle and Hawking, Phys.Rev. D13 (1976)] $P_{out} = e^{-E/T_H} P_{in}$, we have $T_H = \frac{\hbar}{8\pi M}$.

[4] Or simply consider the Boltzmann factor $e^{-E/T}$ related to P_{out}.

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