A new prototype dynamical system with a generalised mechanical potential

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Prototype dynamical systems

- models for dynamical systems: Van der Pol oscillator
- normal forms for bifurcations: Hopf bifurcation
- models for chaos: Lorenz system
- construction methods

[Dangelmayr,G. (1992); Deng, B. (1994); Ucar, A. (2003)]

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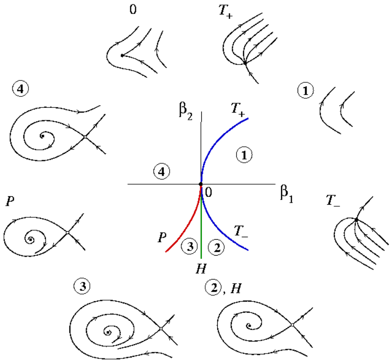
Prototype dynamical systems

- the Bogdanov-Takens normal form:

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = \beta_1 + \beta_2 y_1 + y_1^2 \pm y_1 y_2$$



[Takens, F. (1974), Bogdanov, R. I. (1975)]

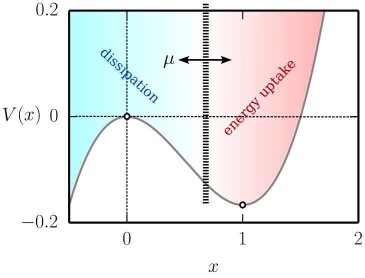
Bogdanov-Takens system revisited

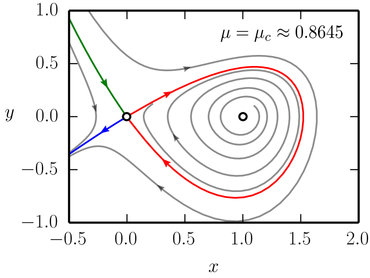
- original B-T system:

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = \beta_1 + \beta_2 y_1 + y_1^2 \pm y_1 y_2$$

- the B-T as a mechanical system ($\beta_1 = 0$, $\beta_2 = 1$):

$$\ddot{x} = (x - \mu)\dot{x} - V'(x) \qquad \dot{x} = y$$
$$V(x) = x^3/3 - x^2/2 \qquad \dot{y} = (x - \mu)y - x^2 + x$$





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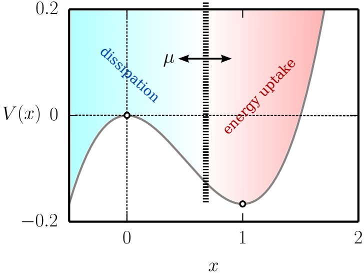
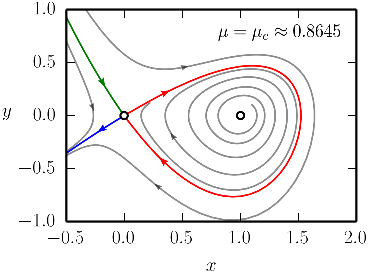
- regions of dissipation and energy uptake:

$$\dot{E} = (x-\mu)y^2 \qquad E = \frac{y^2}{2} + V(x)$$





- fixpoints and dissipation controlled stability:

$$\ddot{x} = (x-\mu)\dot{x} - V'(x) \qquad \dot{x} = y$$

$$V(x) = x^3/3 - x^2/2 \qquad \dot{y} = (x-\mu)y - x^2 + x$$

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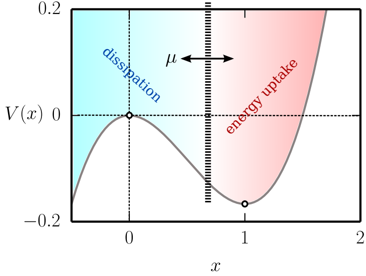
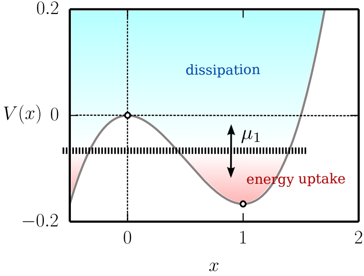
Motivation

- constructing dynamical systems with predefined properties:
  - dimensionality ?
  - fixpoints ?
  - stability ?
  - limit cycles ?
  - bifurcations ?
  - chaos ??
- simple, intuitive ?!

[Deng, B. (1994)]

Generalised friction term





$$\dot{x} = y$$
$$\dot{y} = (x - \mu)y - V'(x)$$

$$\dot{x} = y$$
$$\dot{y} = f(V(x))y - V'(x)$$

- new friction term: $f(V(x)) = \mu_1 - V(x)$

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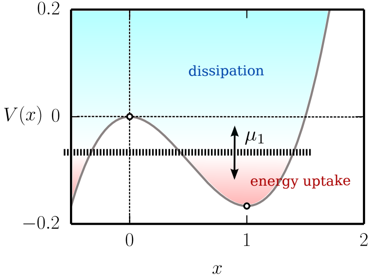
Fixpoints and stability, $d = 1$

- fixpoints at local minima and maxima:

$$\dot{x} = y \qquad\qquad y^* = 0$$
$$\dot{y} = f(V(x))y - V'(x) \qquad\qquad V'(x^*) = 0$$

- stability of fixpoints:

$$J = \begin{pmatrix} 0 & 1 \\ V''(x^*) & f(V(x^*)) \end{pmatrix}$$



$$d = \det(J) = V''(x^*) \Rightarrow \text{saddles}$$
$$t = \mathrm{Tr}(J) \; = f(V(x^*))$$

$$\lambda_{1,2} = \frac{t \pm \sqrt{t^2 - 4d}}{2} \Rightarrow$$

when $f(V(x^*)) \to 0 \;\Rightarrow\; \lambda_{1,2} = \pm i\sqrt{V''(x^*)} \;\Rightarrow$ Hopf bif. at x_n^*

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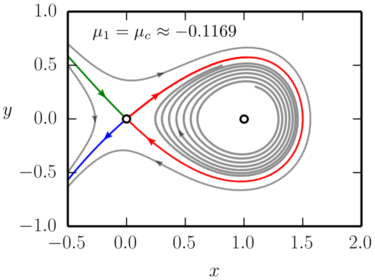

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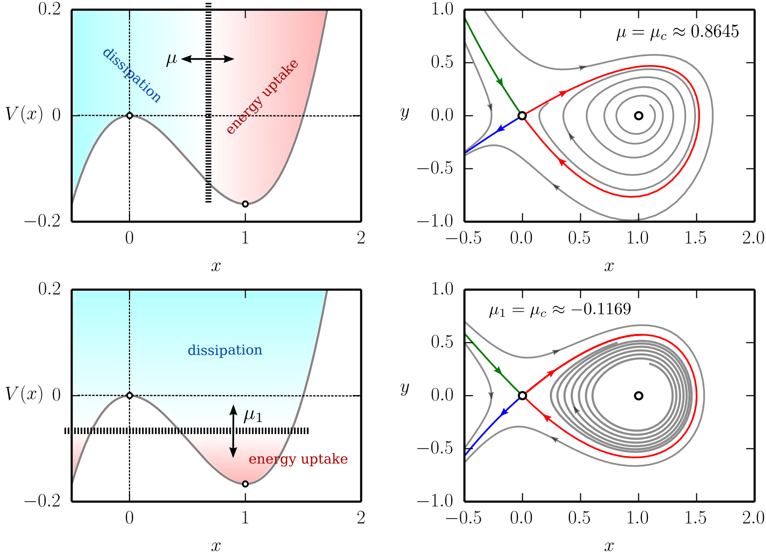
Generalised friction term



Prototype dynamical system

- a new class of $2d$ dimensional prototype systems:

$$\dot{\mathbf{x}} = \mathbf{y}$$
$$\dot{\mathbf{y}} = f(V(\mathbf{x}))\mathbf{y} - \nabla V(\mathbf{x})$$

- d dimensional mechanical system:

$$\ddot{\mathbf{x}} - f(V(\mathbf{x}))\dot{\mathbf{x}} + \nabla V(\mathbf{x}) = 0$$

$$E = \mathbf{y}^2/2 + V(\mathbf{x})$$
$$\dot{E} = f(V(\mathbf{x}))\,\mathbf{y}^2$$

- Liénard system:

$$\ddot{x} + f(x)\dot{x} + g(x) \;=\; 0$$

- example - Van der Pol oscillator:

$$\ddot{x} - \epsilon\left(1 - x^2\right)\dot{x} + x = 0$$

$$f(V) = \epsilon(1 - 2V) \qquad V(x) = \frac{x^2}{2}$$

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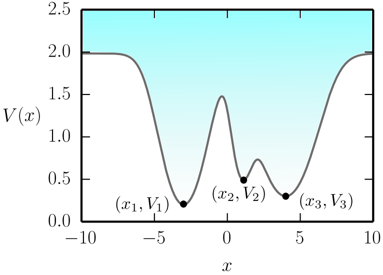
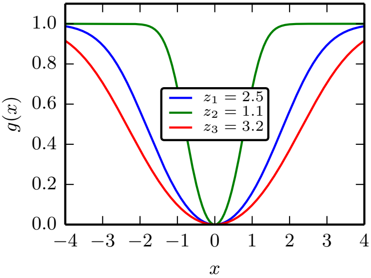
Generalised mechanical potentials

- potentials with a predefined number of local minima
- minima: $\mathbf{x}_n$ coordinate, V_n depth, and z_n half-width

$$V(\mathbf{x}) = \prod_n \left(g_n(\mathbf{x} - \mathbf{x}_n) + \frac{V_n}{p_n} \right) \qquad g_n(\mathbf{z}) = \tanh(\mathbf{z}^2 / {z_n}^2)$$





- coordinates $x_1 = -3$, $x_2 = 1$, $x_3 = 4$
- depths $V_1 = 0.2$, $V_2 = 0.5$, $V_3 = 0.3$

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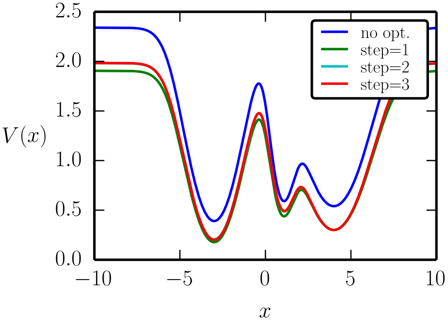
Generalised mechanical potentials

- the p_n parameters are defined self-consistently:

$$p_n = \prod_{m \neq n} \left(g_n(\mathbf{x}_n - \mathbf{x}_m) + \frac{V_m}{p_m} \right)$$

$$V(\mathbf{x}_n) = \frac{V_n}{p_n} \prod_{m \neq n} \left(g_n(\mathbf{x}_n - \mathbf{x}_m) + \frac{V_m}{p_m} \right) = V_n$$



- no optimisation: $p_{1,2,3} = 1$
- after 3 steps: $p_1 = 1.7$, $p_2 = 0.9$, $p_3 = 1.6$

Fixpoints and stability, $d = 1$

- fixpoints at local minima and maxima:

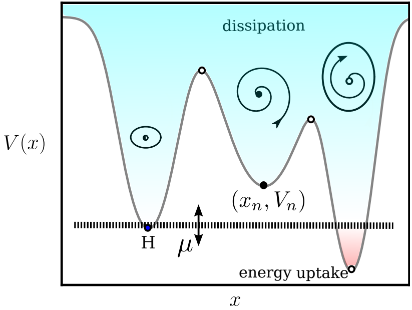
$$\dot{x} = y \qquad\qquad y^* = 0$$
$$\dot{y} = f(V(x))y - V'(x) \qquad\qquad V'(x^*) = 0$$

- stability of fixpoints:

$$J = \begin{pmatrix} 0 & 1 \\ V''(x^*) & f(V(x^*)) \end{pmatrix}$$

$$d = \det(J) = V''(x^*)$$
$$t = \mathrm{Tr}(J) = f(V(x^*))$$

$$\lambda_{1,2} = \frac{t \pm \sqrt{t^2 - 4d}}{2} \Rightarrow$$



when $f(V(x^*)) \to 0 \Rightarrow \lambda_{1,2} = \pm i\sqrt{V''(x^*)} \Rightarrow$ Hopf bif. at x_n^*

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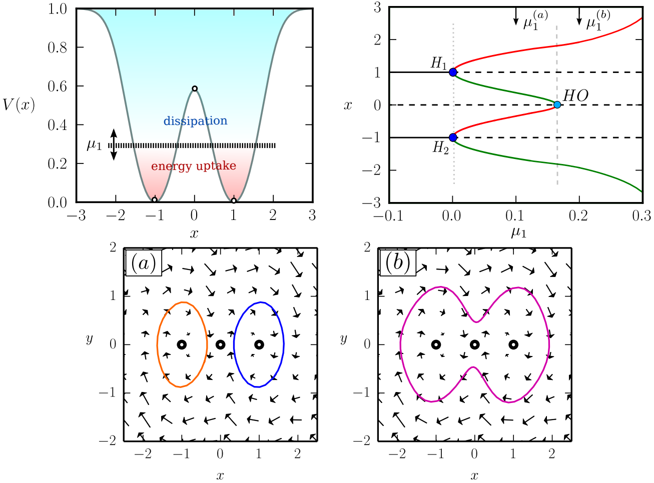
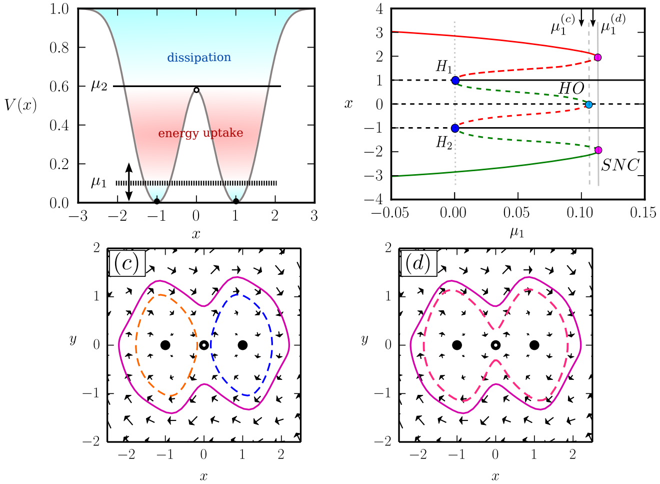
Merging limit cycles



- double well pot.: $x_{1,2} = \pm 1$, $V_{1,2} = 0$, $z_{1,2} = 1$, $p_{1,2} = 1$
- linear friction term: $f_1(V) = -\alpha(V - \mu_1)$ with $\alpha = 1$

Saddle node bifurcation of cycles



- quadratic friction term: $f_2(V) = -\alpha(V - \mu_1)(V - \mu_2)$
- with $\alpha = 5$ and $\mu_2 = 0.6$

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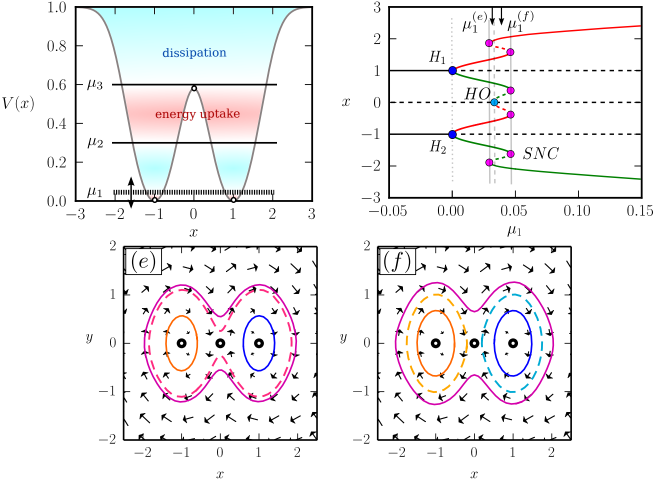
Cascades of limit cycle bifurcations



- cubic friction: $f_3(V) = -\alpha(V - \mu_1)(V - \mu_2)(V - \mu_3)$
- with $\alpha = 5$, $\mu_2 = 0.3$ and $\mu_3 = 0.6$

Fixpoints and stability, general d

- fixpoints at local minima and maxima:

$$\dot{\mathbf{x}} = \mathbf{y} \qquad\qquad \mathbf{y}^* = 0$$

$$\dot{\mathbf{y}} = f(V(\mathbf{x}))\mathbf{y} - \nabla V(\mathbf{x}) \qquad\qquad \nabla V(\mathbf{x}^*) = 0$$

- stability of the $\mathbf{q}^* = (\mathbf{x}^*, \mathbf{y}^*)$ fixpoints:

$$J(\mathbf{q}^*) = \begin{pmatrix} O_d & I_d \\ -H_d & aI_d \end{pmatrix}, \qquad H_d = (H_{i,j}(\mathbf{x}^*)) = \left(\left. \frac{\partial^2 V}{\partial x_i \partial x_j} \right|_{\mathbf{x}^*} \right)$$

- with $a = f(V(\mathbf{x}^*))$ friction term:

$$\det(J - \lambda I_{2d}) = \begin{vmatrix} -\lambda I_d & I_d \\ -H_d & (a-\lambda)I_d \end{vmatrix} = \det(-\lambda(a-\lambda)I_d + H_d) =$$

$$= \det(H_d - \gamma I_d) = \prod_{i=1}^{d} (\gamma - \gamma_i) = 0$$

Fixpoints and stability, general d

- it is enough to know the γ_i eigenvalues of the Hessian

$$\det(J-\lambda I_{2d}) = \det(H_d-\gamma I_d) = \prod_{i=1}^{d}(\gamma-\gamma_i) = 0$$

- since $\gamma = \lambda(a-\lambda) \quad \Rightarrow \quad \lambda_i^{\pm} = \frac{1}{2}(a \pm \sqrt{a^2-4\gamma_i})$
- at local minima of the potential $\gamma_i > 0 \quad \Rightarrow$
  $\lambda_i^{\pm} = \pm i\sqrt{\gamma_i}\,, \qquad \text{when} \qquad a = f(V) \to 0$

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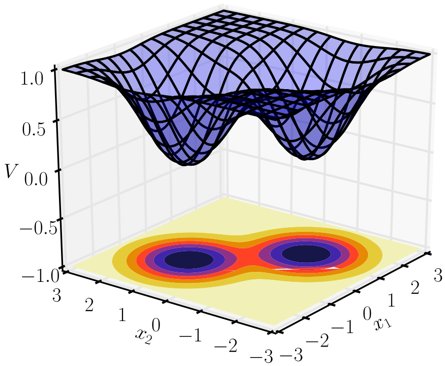
2D double well potential

- symmetric potential function with two minima:

$$V(\mathbf{x}) = g(\mathbf{x}-\mathbf{x}_1)g(\mathbf{x}-\mathbf{x}_2)\,, \qquad g(\mathbf{z}) = \tanh(4\mathbf{z}^2/9)$$



- with: $\mathbf{x}_{1,2} = \pm(1,-1)$, $V_{1,2} = 0$, $z_{1,2} = 1.5$, $p_{1,2} = 1$

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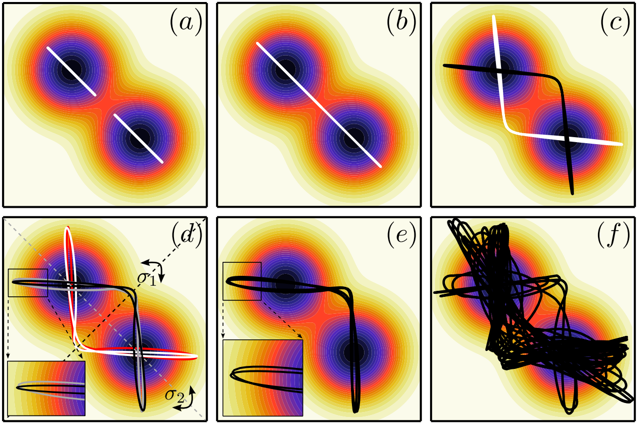
Chaos via period doubling of limit cycles



- linear friction term: $f_1(V) = 0.5(\mu - V)$
- $\sigma_{1,2}$ symmetry operators

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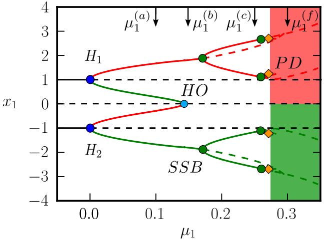

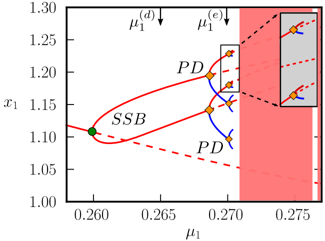

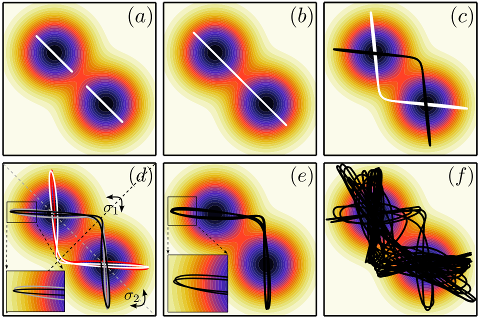

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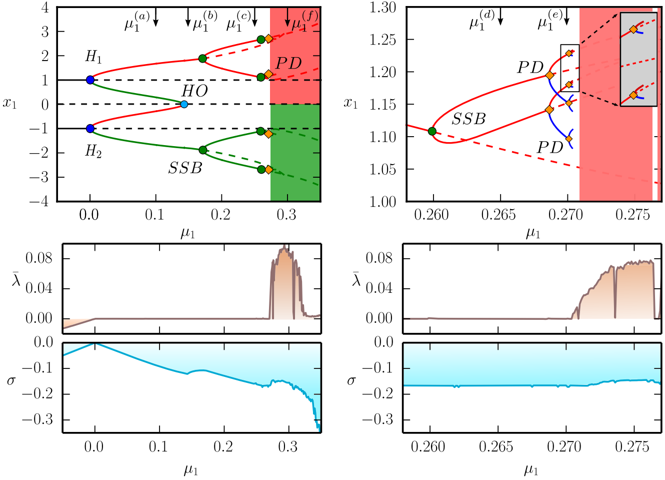

- average Lyapunov exponent: $\bar{\lambda}$
- contraction rate: $\sigma = \left\langle \frac{1}{L} \int_{\Gamma} \nabla \cdot \mathbf{f} \, ds \right\rangle$

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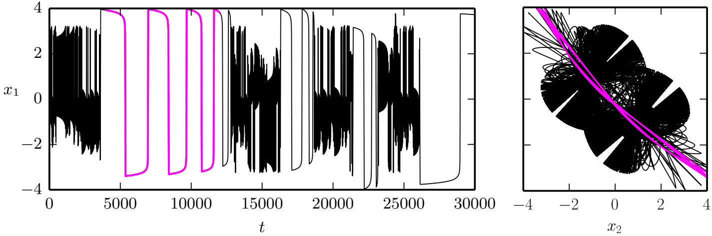
Intermittent chaos



- intermittent dynamics for $\mu_1 = 0.34$

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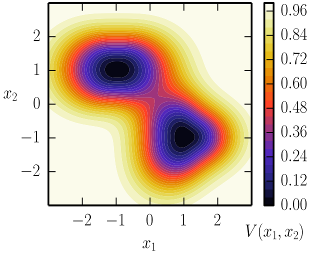
Summary

- new class of Liénard type systems
- cascades of limit cycle bifurcations to chaos
- generalised potentials



- generalised z_n with $\theta(\Delta\mathbf{x}) = \arccos(\Delta x_1 / |\Delta\mathbf{x}|)$

$$z_1^g(\Delta\mathbf{x}) = z_1^s + a_1 \cos(2\theta)$$
$$z_2^g(\Delta\mathbf{x}) = z_2^s + a_2 \cos(\theta) \cdot (\cos(\theta) - 1) \cdot (\cos(\theta) + 1)$$

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