The Impact of Time Delays on Network Synchronization and Coordination in a Noisy Environment

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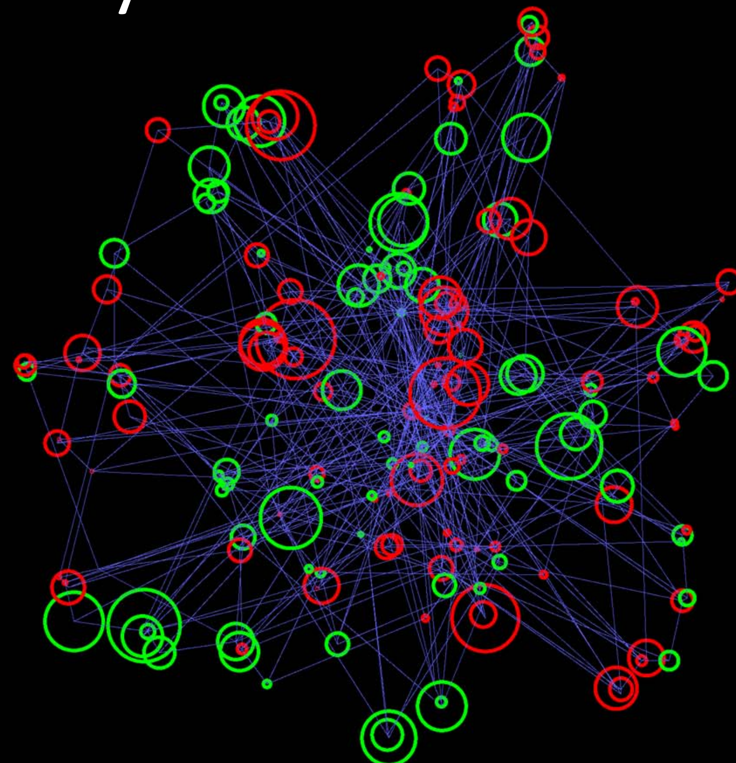

SCNARC
SOCIAL COGNITIVE NETWORKS ACADEMIC RESEARCH CENTER

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Delay Differential Equations

- Macrodynamic theory of business cycles

  Kalecki, 1935,

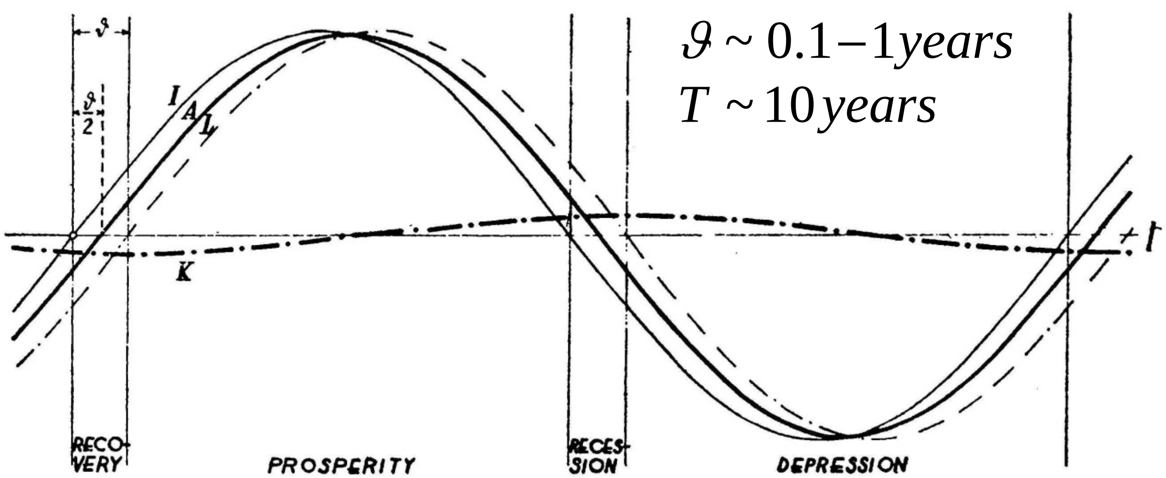
  Frisch & Holme, 1935

$$\frac{dJ}{dt} = aJ(t) - cJ(t - \vartheta)$$

investment orders
(relative to constant demand)

"gestation period"
of an investment

Figure 2 represents the curves of investment orders I, of production of capital goods A, of deliveries of industrial equipment L, and of the volume of industrial equipment K, which correspond to the formulae (38), (39), (40), and (41).



$\vartheta \sim 0.1\text{–}1\,years$

$T \sim 10\,years$

FIGURE 2

Kalecki, *Econometrica* **3**, 327 (1935).

Hutchinson model
(logistic growth with delay in population dynamics)

$$\tau > 0$$

$$\partial_t N(t) = rN(t)\left[1 - \frac{N(t-\tau)}{K}\right]$$

population size

intrinsic growth rate

carrying capacity

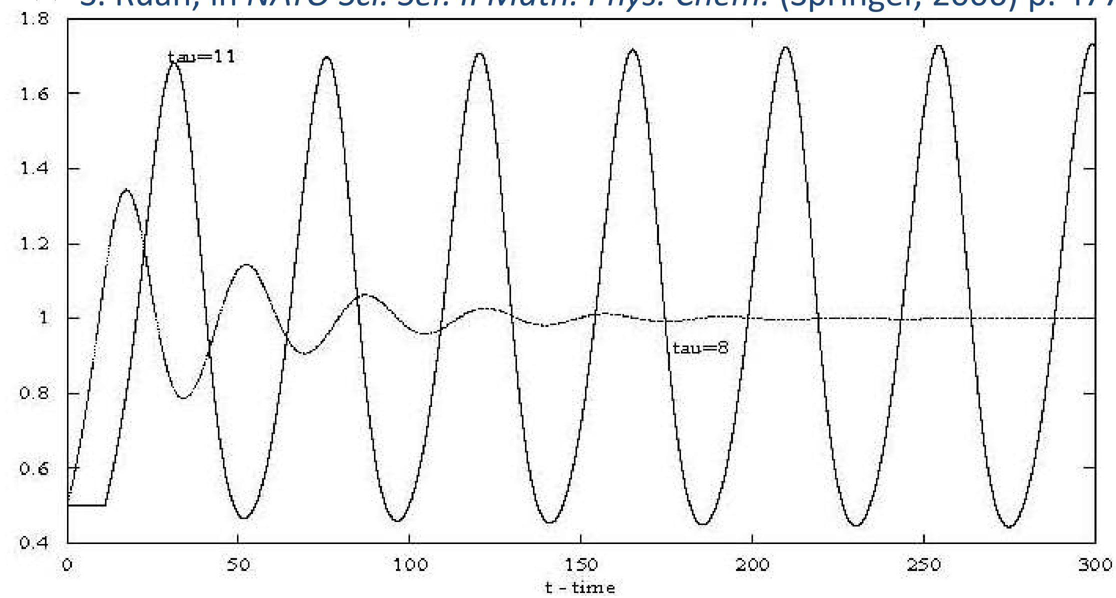
$$(N^* = 0), \quad N^* = K$$

$$N(t) = K + x(t)$$

$$\partial_t x(t) = -rx(t-\tau)$$

stability of $N^* = K$:
$r\tau < \pi/2$

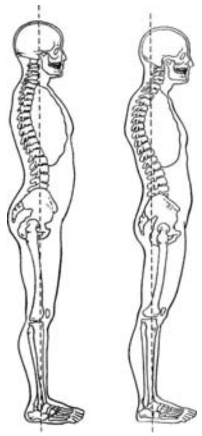
$N(t)$ S. Ruan, in *NATO Sci. Ser. II Math. Phys. Chem.* (Springer, 2006) p. 477

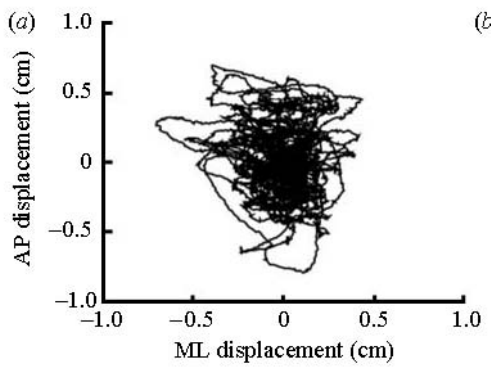


Hutchinson (1948); Maynard Smith (1971); R.M. May (1973)

Balancing (noise, feedback, delay, coordination)

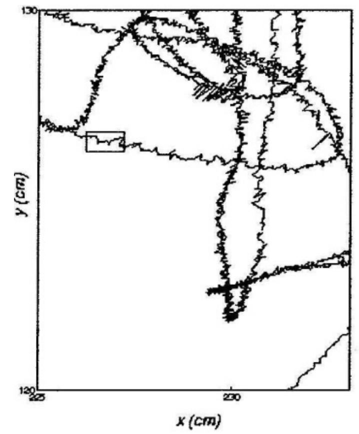
postural sway
[fluctuations in the center of pressure]





Milton *et al.*, *PTRSA* (2011); EPL (2008).

stick balancing at a fingertip



Cabrera *et al.*, *PRL (2002); FNL (2004); CMP* (2006);

Stépán & Kollár, *MCM* (2000).

$$\partial_t h(t) = F[h(t-\tau), \eta(t)]$$

(biological/neurological systems: switch-type/discontinuous/threshold control)

Synchronization/Coordination in Coupled Systems

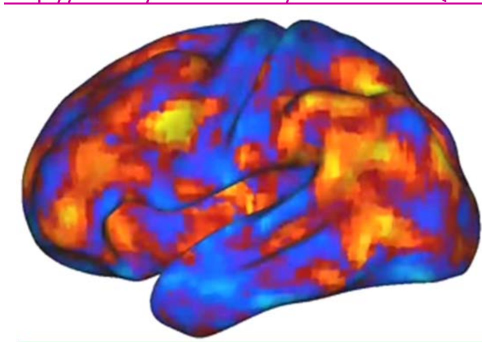
- individual units or agents (represented by static or mobile nodes) attempt to adjust their local state variables (e.g., pace, load, alignment, coordination) in a decentralized fashion.

  Craig Reynolds (1987); Vicsek *et al*. (1995); Cavagna *et al*. (2010).

- nodes interact or communicate only with their local neighbors in the network, possibly to improve global performance or coordination.
- nodes react (perform corrective actions) to the information or signal received from their neighbors possibly with some time lag (as result of finite transmission, queuing, processing, or execution delays)
- Applications: autonomous coordination, unmanned aerial vehicles, microsatellite clusters, sensor and communication networks, load balancing, flocking, distributed decision making in social networks

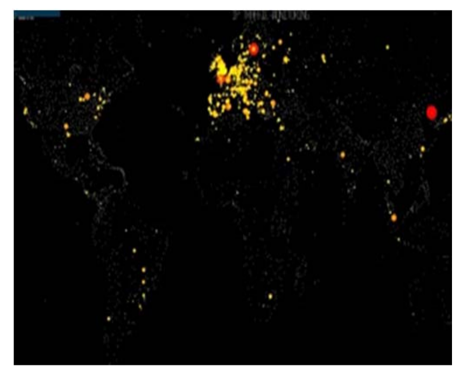
flocking birds

http://www.youtube.com/watch?v=VaQ66lDZ-08



spontaneous brain activity (fMRI)
(Justin Vincent; http://martinos.org/~vincent/)

http://www.youtube.com/watch?v=6AmSpHxnKm8

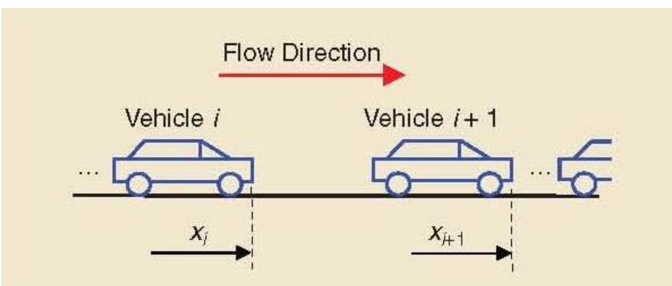


IP activity
(Zeus load balancer)

Delays in Microscopic Vehicular Flow

Human drivers have ***reaction delays (awareness, decision, and execution delays)***, which, in part, depend on the ***drivers' cognitive and physiological states***

Y. Sugiyama, M. Fukui, M. Kikuchi, K. Hasebe, A. Nakayama, K. Nishinari, S.-i. Tadaki, S. Yukawa, *New Journal of Physics* **10** 033001 (2008); http://dx.doi.org/10.1088/1367-2630/10/3/033001 .
Shockwave traffic jam recreated for first time, *New Scientist*,2008);
http://www.newscientist.com/article/dn13402
http://www.youtube.com/watch?v=Suugn-p5C1M



Sipahi et al., "Stability and Stabilization of Systems with Time Delay", IEEE Control Systems Magazine (2011);
http://dx.doi.org/10.1109/MCS.2010.939135

$$\dot{v}_i(t) = \kappa[v_{i+1}(t-\tau) - v_i(t-\tau)]$$

G. Orosz, E. Wilson, and G. Stépán (Eds.), "Traffic jams: dynamics and control", *Philos. Trans . R. Soc. A* **368**, 4455 (2010).
G. Orosz and G. Stépán, "Subcritical Hopf bifurcations in a car-following model with reaction-time delay", *Proc. R. Soc. A* **462** 2643 (2006).
D. Helbing, "Traffic and related self-driven many-particle systems," *Rev. Mod. Phys*. **73**, 1067, (2001).

Synchronization/Coordination in Networks

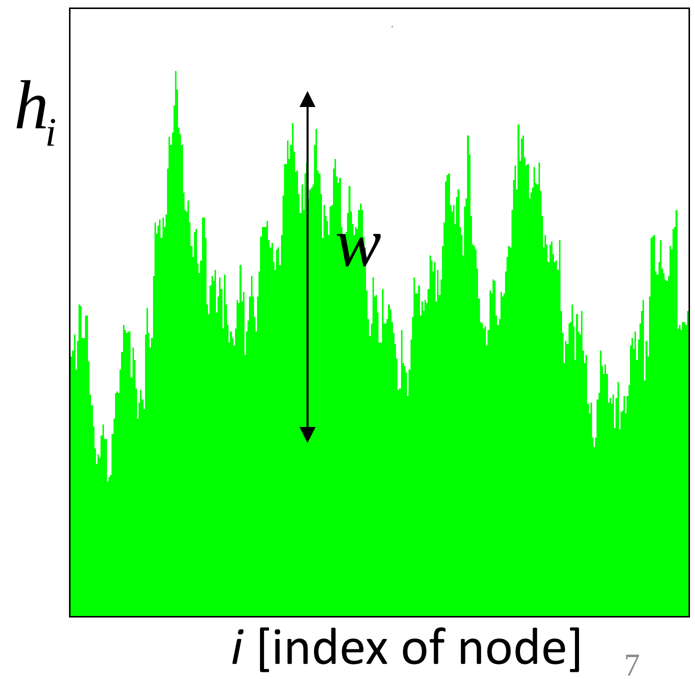
- $h_i(t)$: local state variable
- a measure of **synchronization, coordination**, or **load balancing** efficiency:

the spread of the synchronization landscape, w:

$$\left\langle w^2(t) \right\rangle = \left\langle \frac{1}{N} \sum_{i=1}^{N} [h_i(t) - \bar{h}(t)]^2 \right\rangle$$

$$\bar{h}(t) = \sum_l h_l(t)$$

synchronizability: $\left\langle w^2(\infty) \right\rangle < \infty$



Synchronization/Coordination in a Noisy Environment with Time Delays

$$\partial_t h_i(t) = -\sum_j C_{ij}[h_i(t-\tau) - h_j(t-\tau)] + \eta_i(t)$$

network/coupling strength — delay — noise

$$\partial_t h_i(t) = -\sum_j \Gamma_{ij} h_j(t-\tau) + \eta_i(t)$$

network Laplacian:

$$\Gamma_{ij} = \delta_{ij} C_i - C_{ij}$$

$$\partial_t \tilde{h}_k(t) = -\lambda_k \tilde{h}_k(t-\tau) + \tilde{\eta}_k(t)$$

$$\langle w^2(t) \rangle = \frac{1}{N} \sum_{k=1}^{N-1} \langle \tilde{h}_k^2(t) \rangle$$

eigenvalues: $0 = \lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots \le \lambda_{N-1} = \lambda_{\max}$

Coordination, Noise, Time Delay

$$\partial_t h(t) = -\lambda h(t-\tau) + \eta(t)$$

$$\langle \eta(t)\eta(t')\rangle = 2D\delta(t-t')$$

coupling strength (λ), delay (τ), noise ($\eta(t)$)

- Other applications: **stochastic model for TCP Congestion Window**
  - Misra, Gong, and Towsley (1999), T. Ott and J. Swanson (2006); T. Ott (2006)

- **deterministic**: Frisch & Holme (1935); Hayes (1950); Hutchinson (1948); Maynard Smith (1971); R.M. May (1973)
- **stochastic**: Küchler and Mensch, *SSR* **40**, 23 (1992); Ohira & Yamane, *PRE* (2000); Frank & Beek, *PRE* (2001); Hunt, Korniss, Szymanski, *PRL* (2010)

Coordination, Noise, Time Delay

$$\partial_t h(t) = -\lambda h(t-\tau) + \eta(t)$$

$$\langle \eta(t)\eta(t')\rangle = 2D\delta(t-t')$$

characteristic equation: $(h(t) = ce^{st})$

$$g(s) \equiv s + \lambda e^{-\tau s} = 0$$

$$s_\alpha = s_\alpha(\lambda, \tau), \quad \alpha = 1, 2, \ldots$$

infinitely many relaxation "rates", $\{s_\alpha\}$, for $\tau > 0$

$$\langle h^2(t)\rangle = \sum_{\alpha,\beta} \frac{-2D[1 - e^{(s_\alpha + s_\beta)t}]}{g'(s_\alpha)g'(s_\beta)(s_\alpha + s_\beta)}$$

synchronizability: $\langle h^2(\infty)\rangle < \infty$

synchronizability condition:

$$\mathrm{Re}(s_\alpha) < 0 \quad \forall \alpha$$

⇓

$$\lambda\tau < \pi/2$$

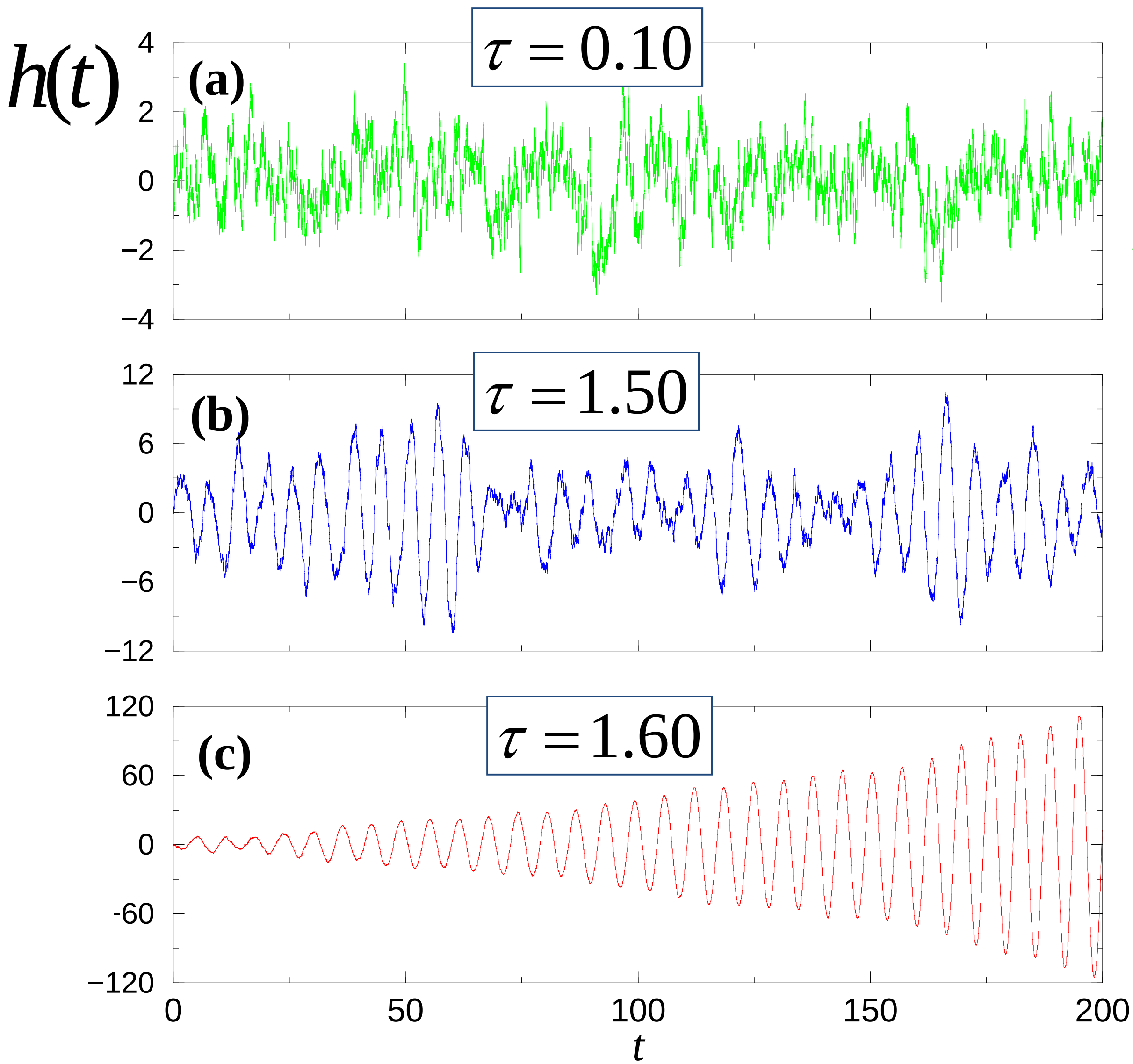
Frisch & Holme (1935); Hayes (1950); Hutchinson (1948); Maynard Smith (1971); R.M. May (1973)

Coordination, Noise, Time Delay

$$(\lambda\tau)_c = \pi / 2$$

$$\lambda = 1, D = 1, dt = 0.01$$

$$\tau_c = \pi / 2 \approx 1.57$$



$$\lambda\tau < \frac{1}{e}$$

$$\frac{1}{e} < \lambda\tau < \frac{\pi}{2}$$

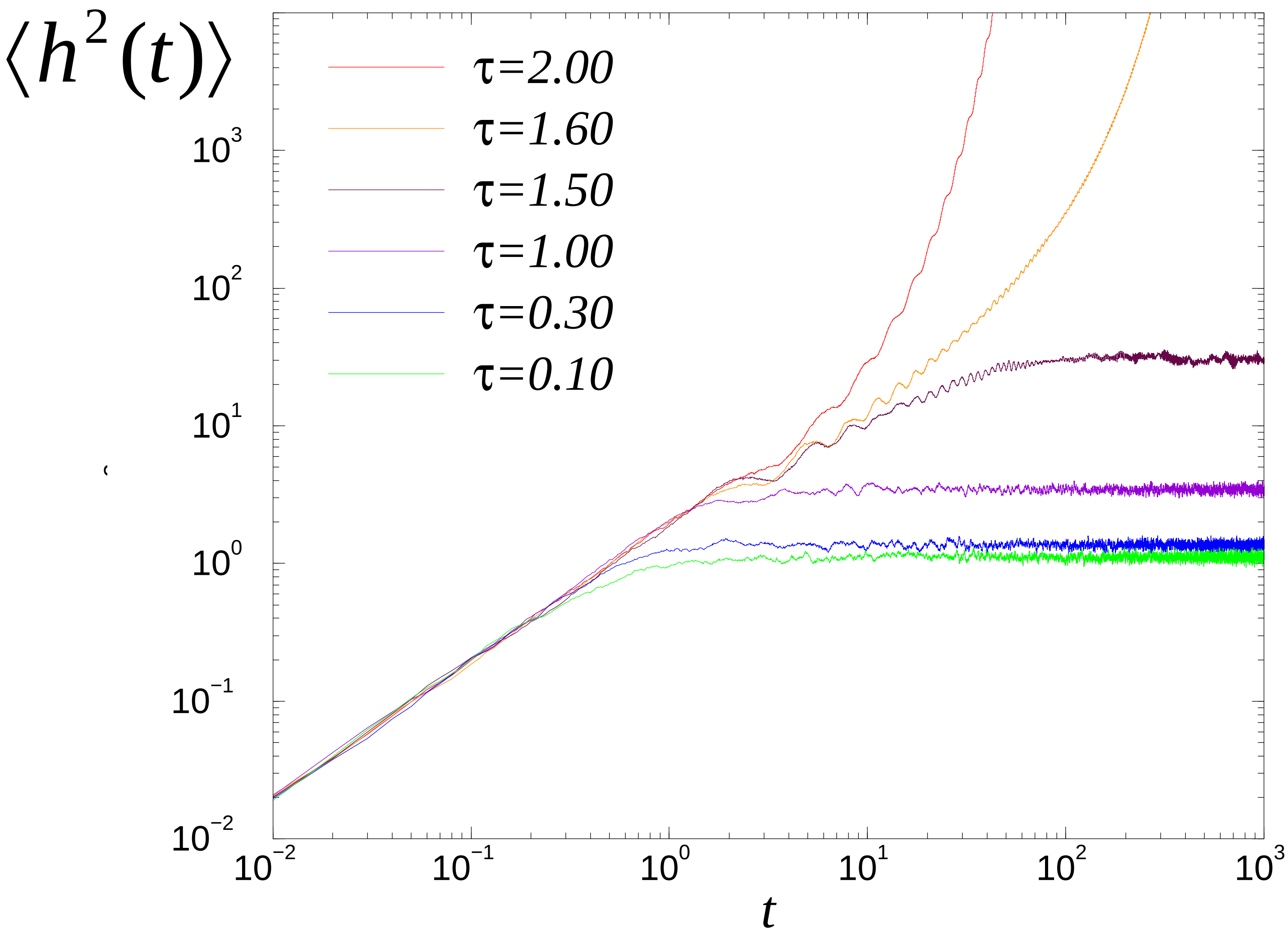
$$\frac{\pi}{2} < \lambda\tau$$

Coordination, Noise, Time Delay

$$(\lambda\tau)_c = \pi / 2$$

$$\lambda = 1, D = 1, dt = 0.01$$

$$\tau_c = \pi / 2 \approx 1.57$$

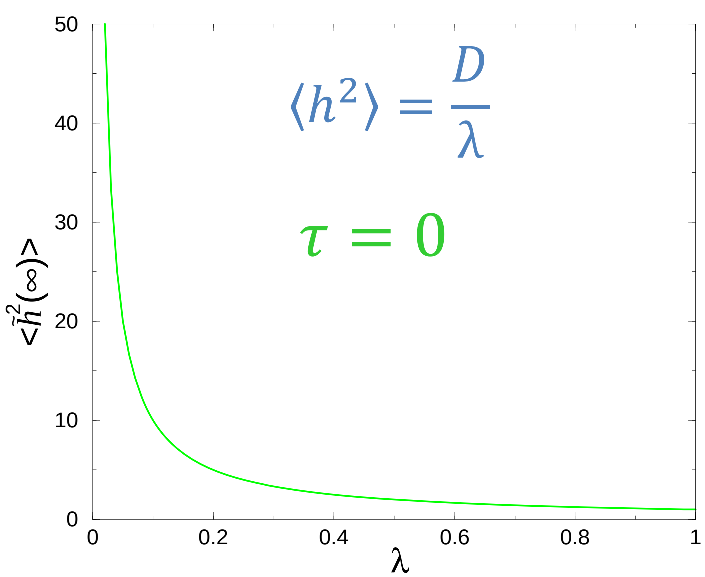


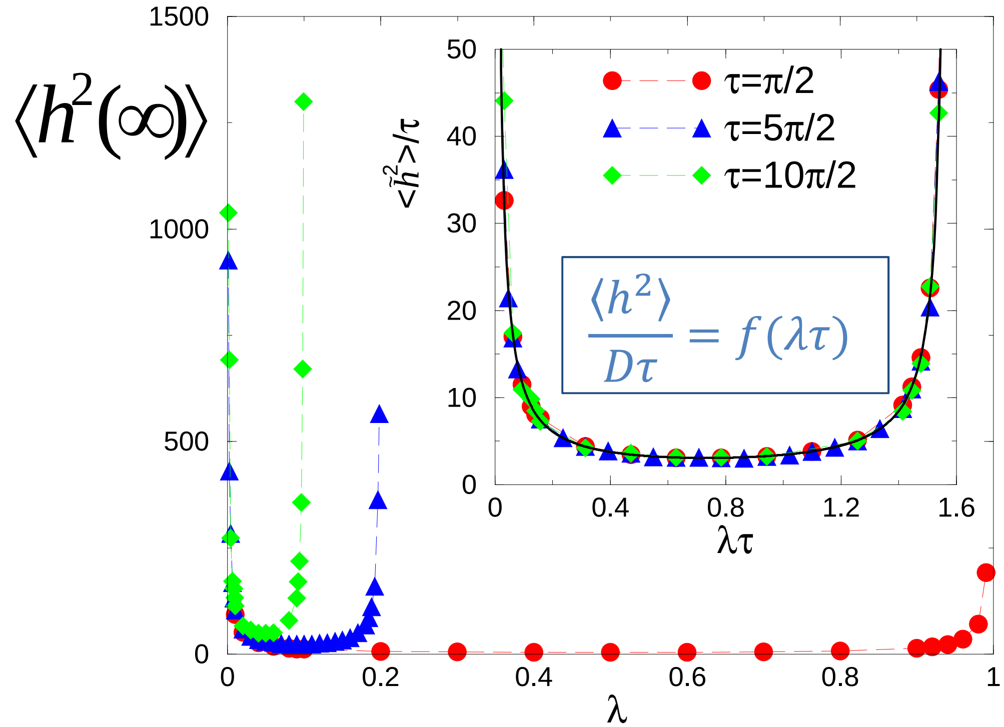
Scaling in the Synchronizable Regime

steady state: $0 < \lambda\tau < \pi/2$

Küchler and Mensch, *SSR* **40**, 23 (1992).

$$\langle h^2 \rangle = D\frac{1+\sin(\lambda\tau)}{\lambda\cos(\lambda\tau)} = D\tau\frac{1+\sin(\lambda\tau)}{\lambda\tau\cos(\lambda\tau)} = D\tau f(\lambda\tau)$$



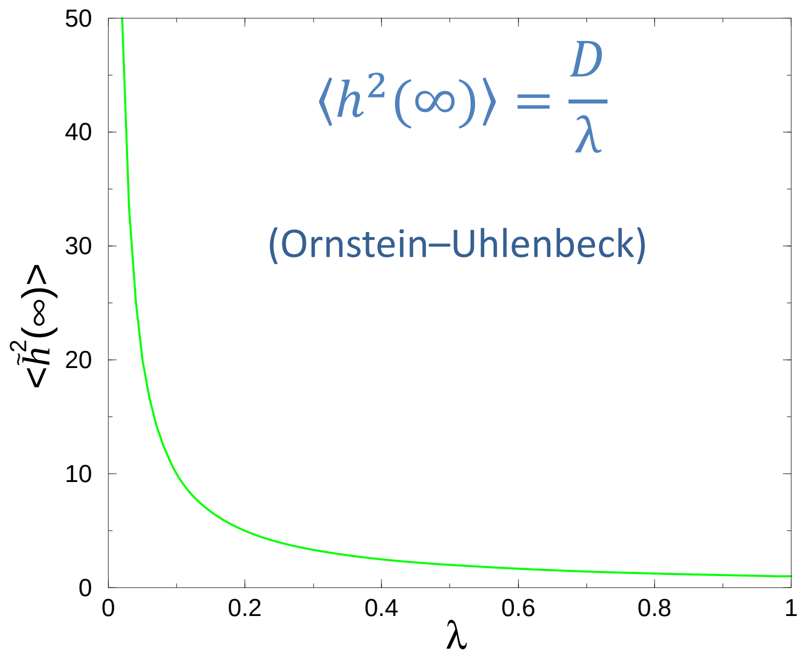


Scaling in the Synchronizable Regime

$$\partial_t h(t) = -\lambda h(t-\tau) + \eta(t)$$

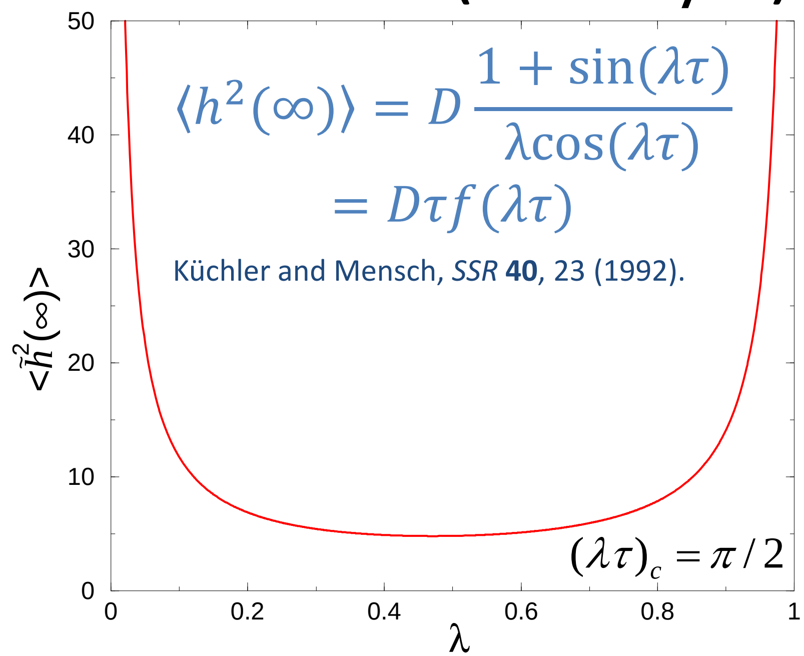
$$\langle \eta(t)\eta(t')\rangle = 2D\delta(t-t')$$

$\tau = 0$



monotonically decreasing *function of the coupling* $\boldsymbol{\lambda}$

$\tau > 0 \quad (\tau = \pi/2)$



non-monotonic *function of the coupling* $\boldsymbol{\lambda}$

Implications for Networks:

$$\left\langle w^2(t)\right\rangle = \frac{1}{N}\sum_{k=1}^{N-1}\left\langle \tilde{h}_k^2(t)\right\rangle = \frac{D\tau}{N}\sum_{k=1}^{N-1} f(\lambda_k\tau)$$

Hunt *et al.*, *PRL* (2010)

Synchronizability and Coordination:

$$\langle \tilde{h}_k^2(\infty)\rangle < \infty \quad \forall k$$

$$\lambda_k \tau < \pi / 2 \quad \forall k$$

$$\lambda_{\max}\tau < \pi / 2$$

Olfati-Saber and Murray (2004)
(deterministic consensus problems)

Limitations of Network Synchronization

Simple example: ***unweighted graphs*** $(C_{ij} = A_{ij})$

$$\frac{N}{N-1} k_{\max} \leq \lambda_{\max} \leq 2k_{\max}, \qquad \lambda_{\max} = O(k_{\max})$$

Fiedler (1973); Anderson and Morley (1985); Mohar (1991)

largest degree

largest eigenvalue of the network Laplacian

Limitations of Network Synchronization

Simple example: ***unweighted graphs*** $(C_{ij} = A_{ij})$

$$\frac{N}{N-1} k_{\max} \le \lambda_{\max} \le 2k_{\max}, \qquad \lambda_{\max} = O(k_{\max})$$

Fiedler (1973); Anderson and Morley (1985); Mohar (1991)

$k_{\max}\tau < \pi/4:$ sufficient for synchronizability/stability

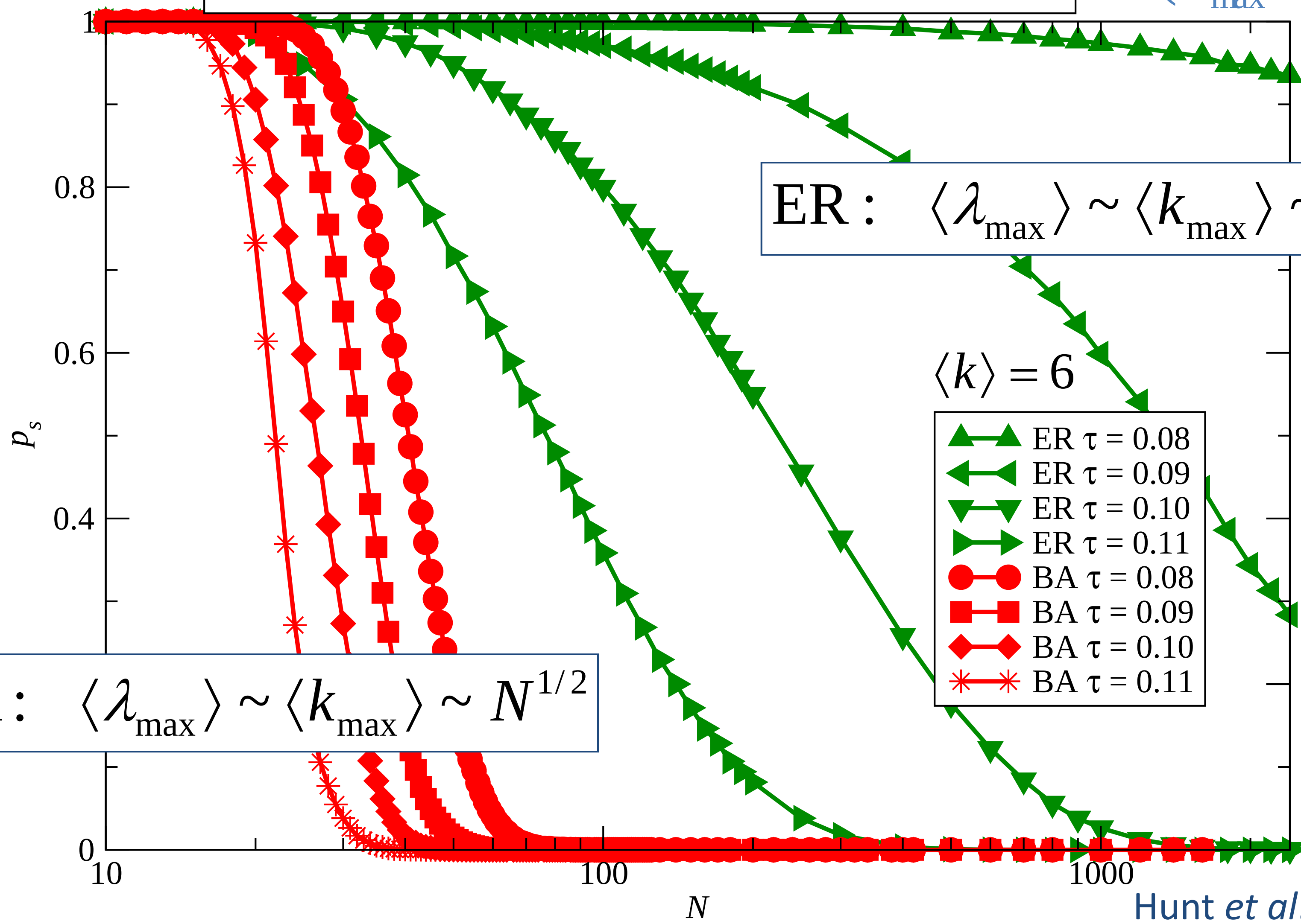
$k_{\max}\tau > \pi/2:$ synchronization/stability breaks down

- networks with potentially large degrees can be extremely vulnerable to intrinsic network delays while attempting to synchronize, coordinate, or balance their tasks, load, etc.

Limitations of Network Synchronization

heterogeneous vs. homogeneous random graphs

$p^s(\tau, N)$ fraction of synchronizable networks $(\lambda_{max}\tau < \pi/2)$



$\text{ER}: \quad \langle \lambda_{max} \rangle \sim \langle k_{max} \rangle \sim \ln(N)$

$\text{BA}: \quad \langle \lambda_{max} \rangle \sim \langle k_{max} \rangle \sim N^{1/2}$

Hunt *et al.*, *PRL* (2010)

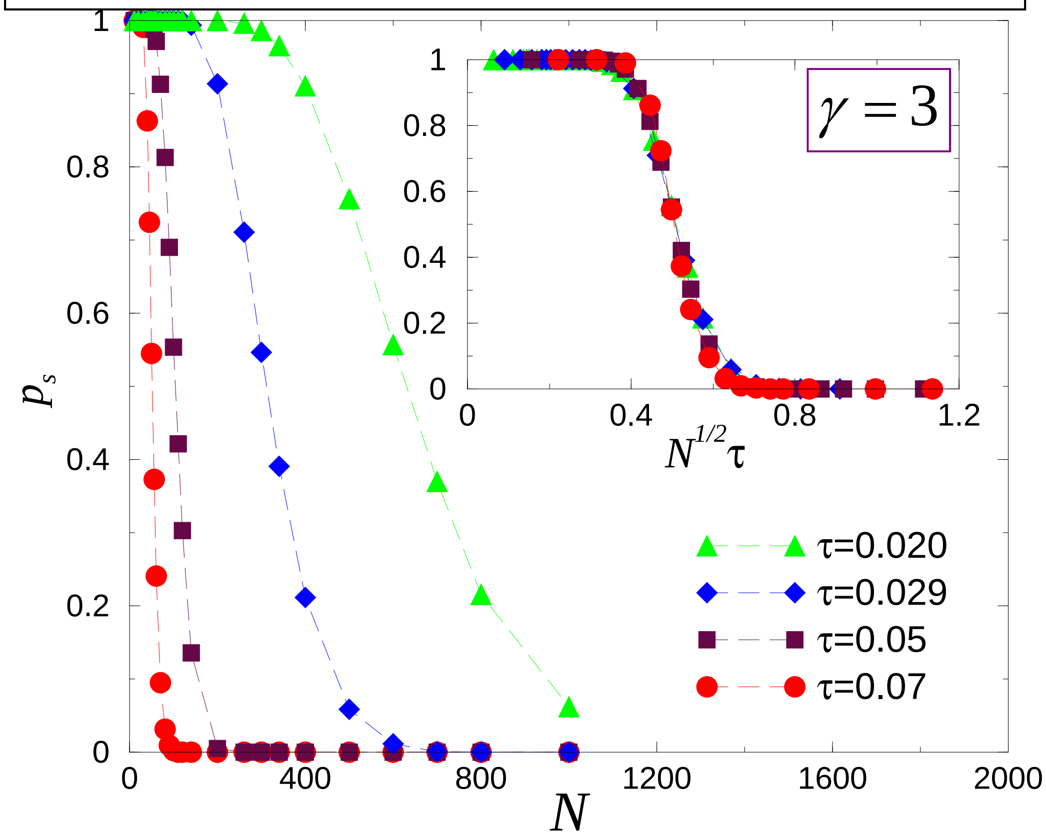
Limitations of Network Synchronization

Example: scale-free (BA) network ensemble with a natural cut-off

$$P(k) \propto k^{-\gamma}$$

$$\langle \lambda_{\max} \rangle \sim \langle k_{\max} \rangle \sim N^{1/(\gamma-1)}$$

$p^s(\tau, N)$ fraction of synchronizable networks



$$p^s(\tau, N) = P_N^{\max<}(\pi / 2\tau) = g(\tau N^{1/(\gamma-1)})$$

synchronization and coordination breaks down for:

$$\tau N^{1/(\gamma-1)} >> 1$$

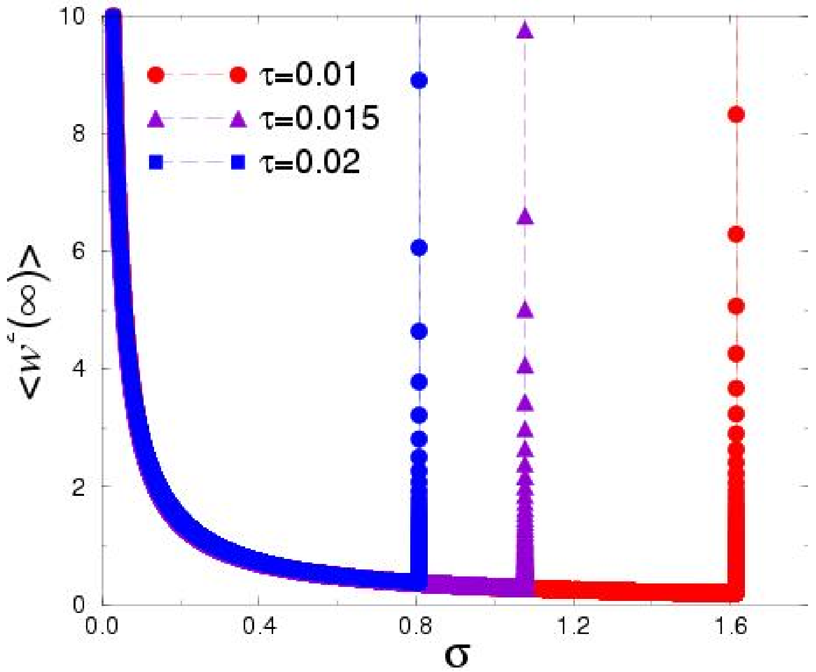
Hunt *et al.*, *PRL* (2010)

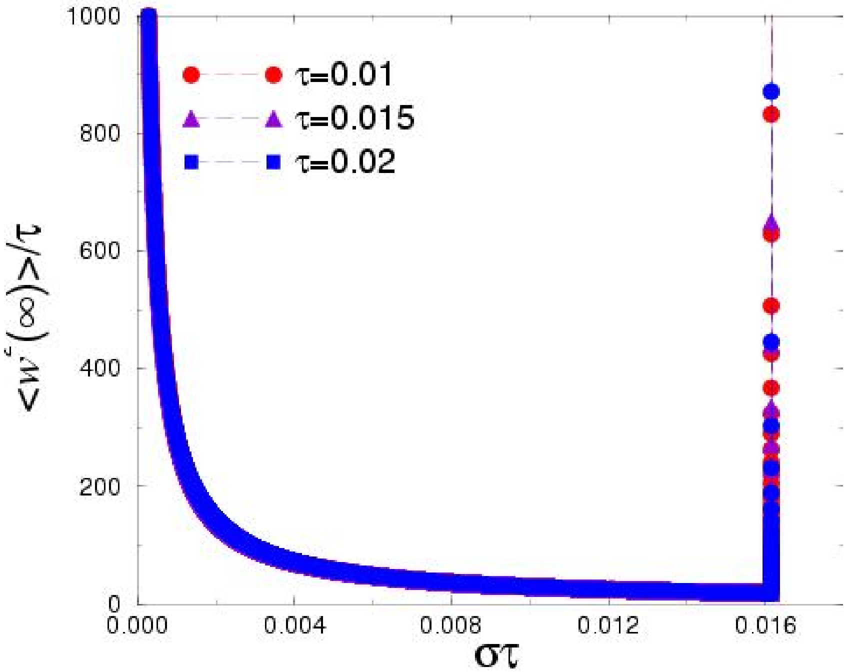
Scaling in the Synchronizable Regime with Uniform Coupling Strength

$$C_{ij} = \sigma A_{ij} \quad \rightarrow \quad \lambda_k' = \sigma \lambda_k$$

$$\left\langle w^2(\infty) \right\rangle_{\sigma,\tau} = \frac{D\tau}{N} \sum_{k=1}^{N-1} f(\sigma\lambda_k\tau) = \tau g(\sigma\tau)$$

$$\frac{\left\langle w^2(\infty) \right\rangle_{\sigma,\tau}}{\tau} = g(\sigma\tau) \qquad (D=1)$$





BA network, N=1000, $\langle k \rangle \approx 6$

Hunt *et al.*, *PRE* (2012)

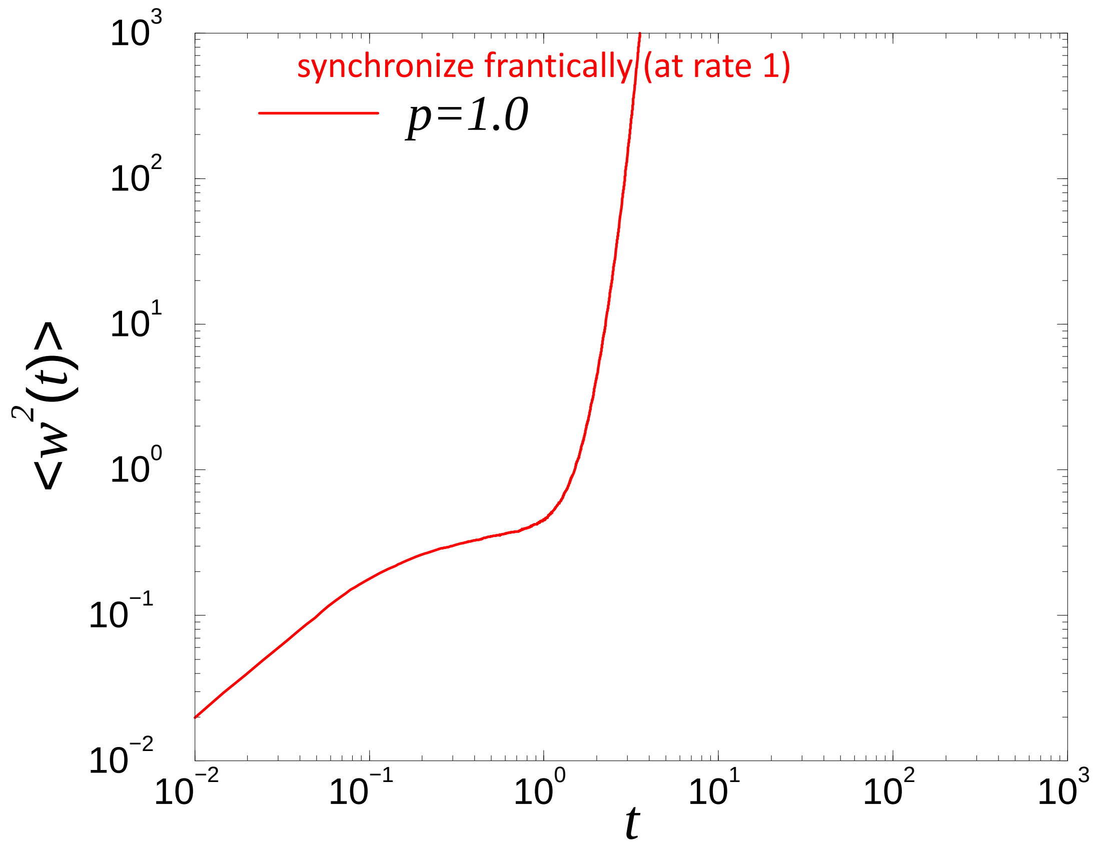
Trade-Offs

$$\partial_t h_i(t) = -\sum_j C_{ij}[h_i(t-\tau) - h_j(t-\tau)] + \eta_i(t)$$

perform local synchronization with rate p

$\lambda_{max}\tau \approx 1.2\pi/2$

BA network, N=200, $\langle k \rangle \approx 6$



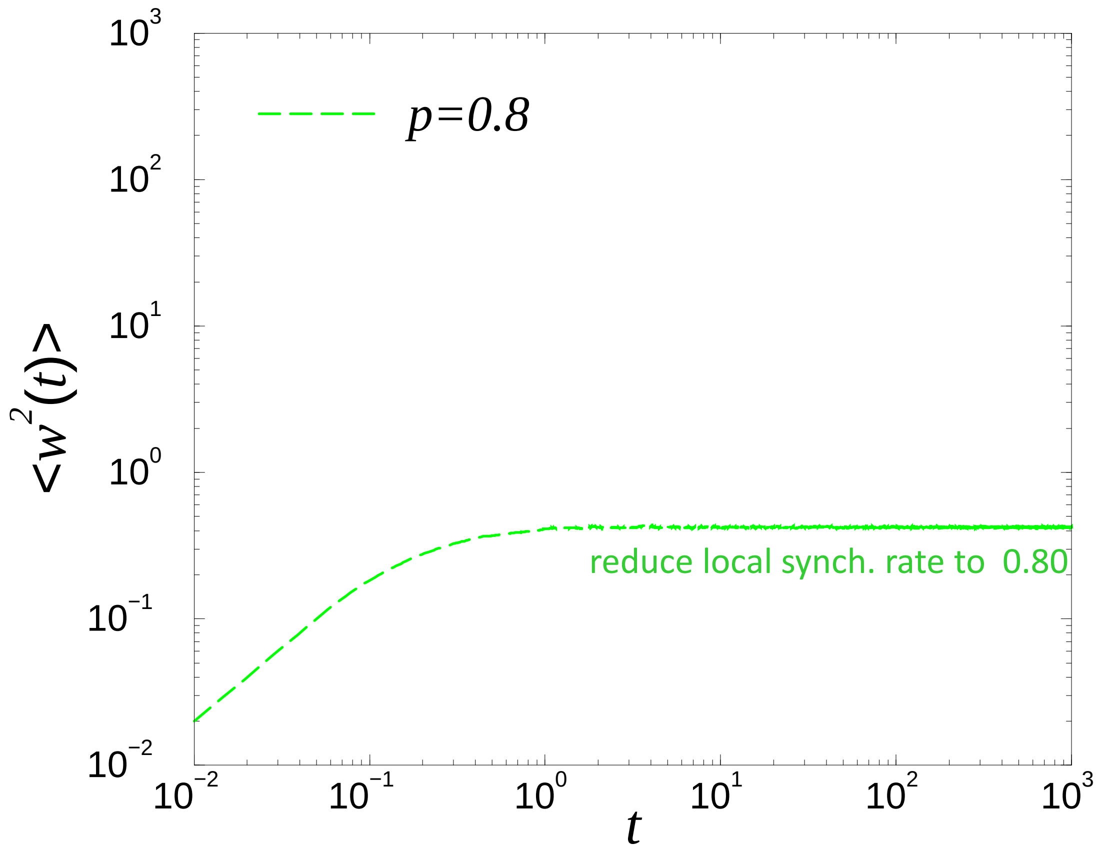
Hunt *et al.*, *PRL* (2010)

Trade-Offs

$$\partial_t h_i(t) = -\sum_j C_{ij}[h_i(t-\tau) - h_j(t-\tau)] + \eta_i(t)$$

BA network, *N*=200, ⟨*k*⟩≈6

$$\lambda_{\max}\tau \approx 1.2\pi/2$$



perform local synchronization with rate *p*

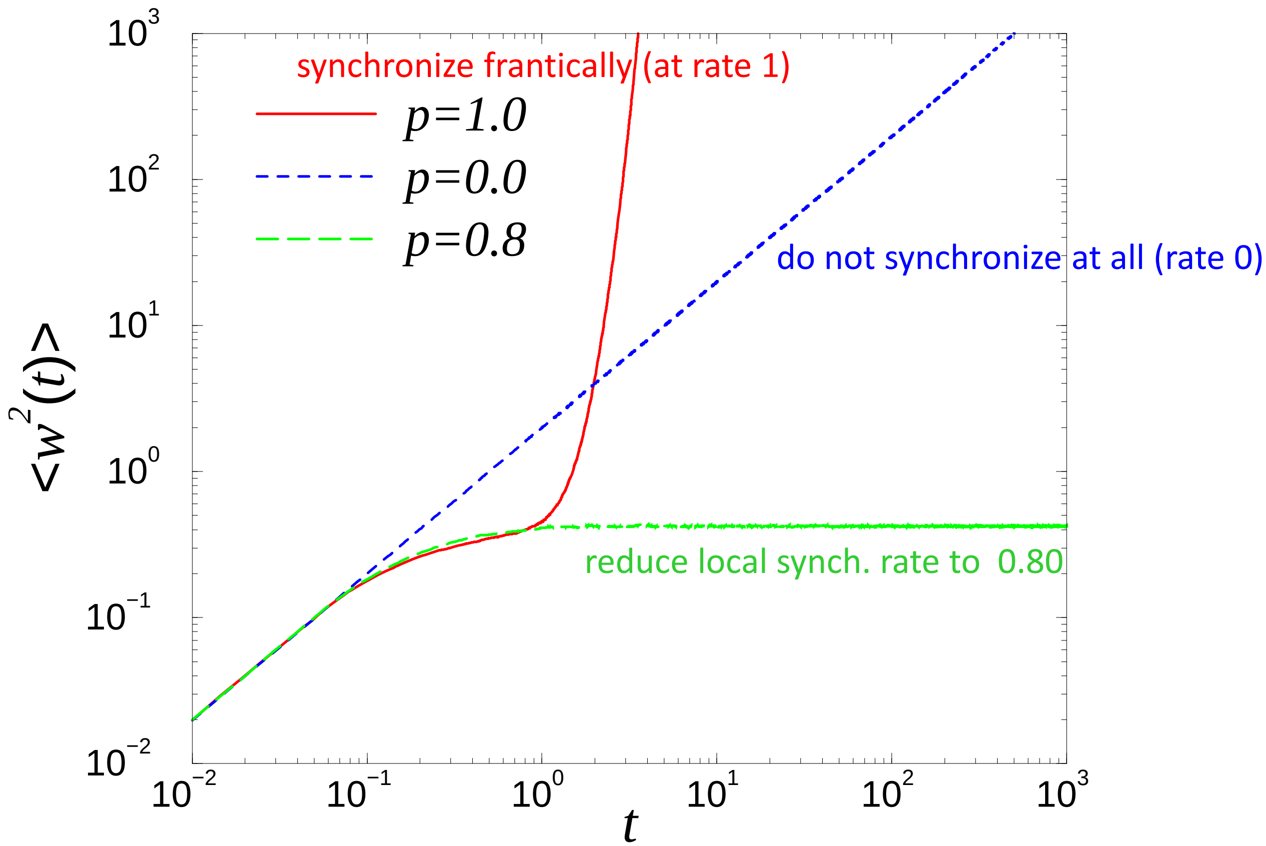
Hunt *et al.*, *PRL* (2010)

Trade-Offs

$$\partial_t h_i(t) = -\sum_j C_{ij}[h_i(t-\tau) - h_j(t-\tau)] + \eta_i(t)$$

BA network, N=200, $\langle k \rangle \approx 6$

$$\lambda_{max}\tau \approx 1.2\pi/2$$



perform local synchronization with rate p

- reducing the local synch. frequency can stabilize the system

(in fact, ***even no synchronization at all is better than "over-synchronization":*** power-law divergence *vs* exp. divergence of the fluctuations with time)

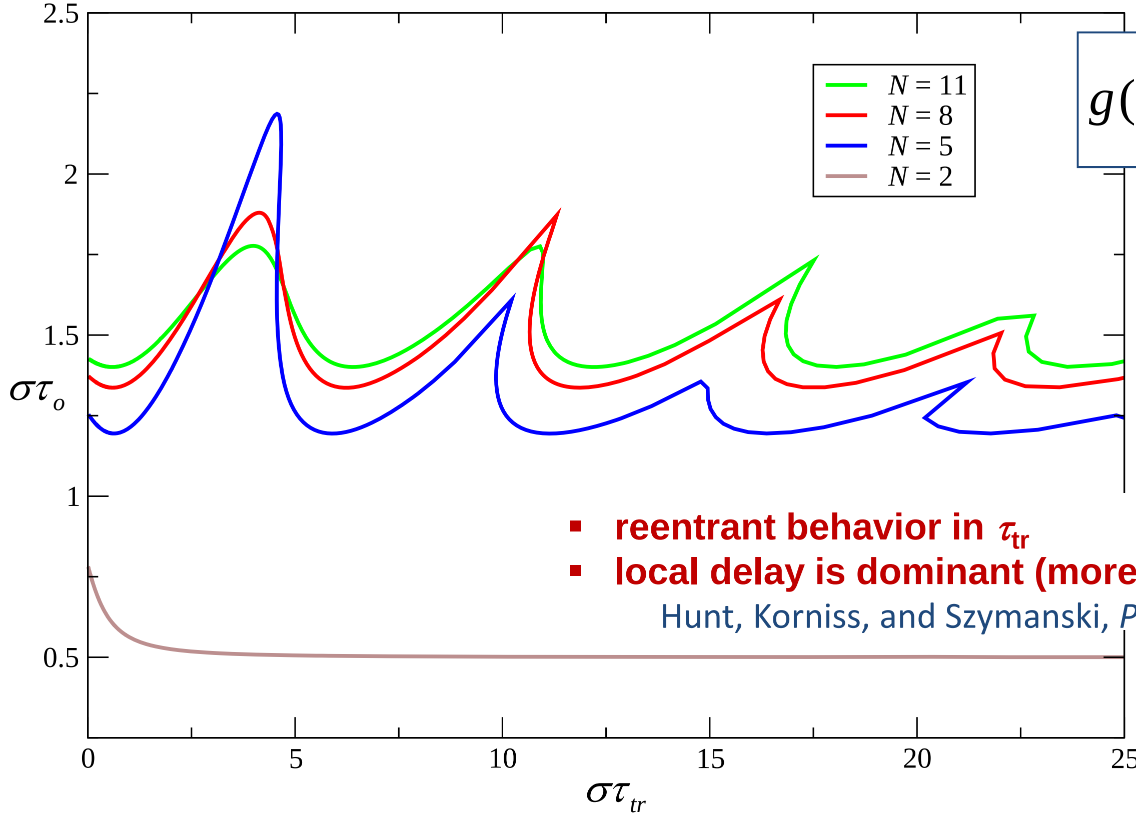
Hunt *et al.*, *PRL* (2010)

Phase Boundary for Competing Time Delays

Complete Graph with *N* nodes ("normalized"):

$$\partial_t \tilde{h}_k(t) = -\sigma \tilde{h}_k(t-\tau_o) - \frac{\sigma}{N-1}\tilde{h}_k(t-\tau_o-\tau_{tr}) + \tilde{\eta}_k(t)$$

$$g(s) \equiv s + \sigma e^{-\tau_o s} + \frac{\sigma}{N-1} e^{-(\tau_o+\tau_{tr})s} = 0$$

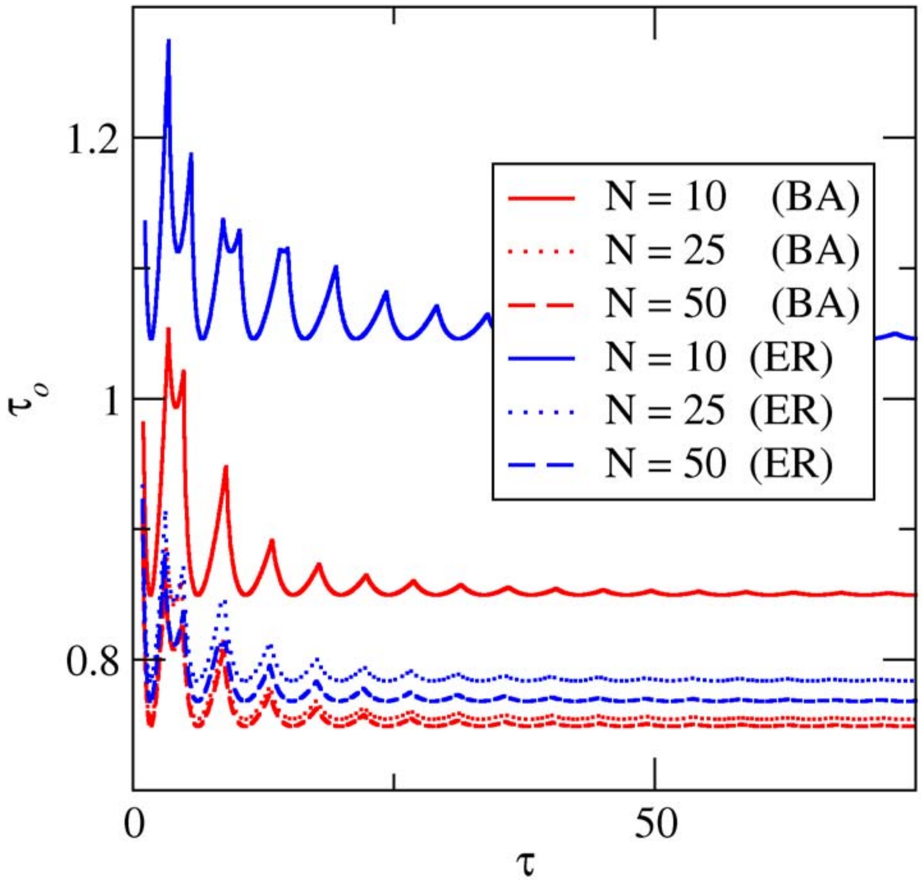


- **reentrant behavior in τ_{tr}**
- **local delay is dominant (more harmful)**

Hunt, Korniss, and Szymanski, *PLA* (2011).

Synchronization and Coordination with Multiple Time Delays

$$\partial_t h_i(t) = -\frac{\sigma}{k_i}\sum_j A_{ij}[h_i(t-\tau_o) - h_j(t-\tau)] + \eta_i(t)$$



τ_o : local delays (reaction, decision, execution)

τ : local delays + transmission, queuing delays

- **local delay is dominant (more harmful)**

Hunt *et al.*, *PRE* (2012).

Global vs. Local Weighted Coupling

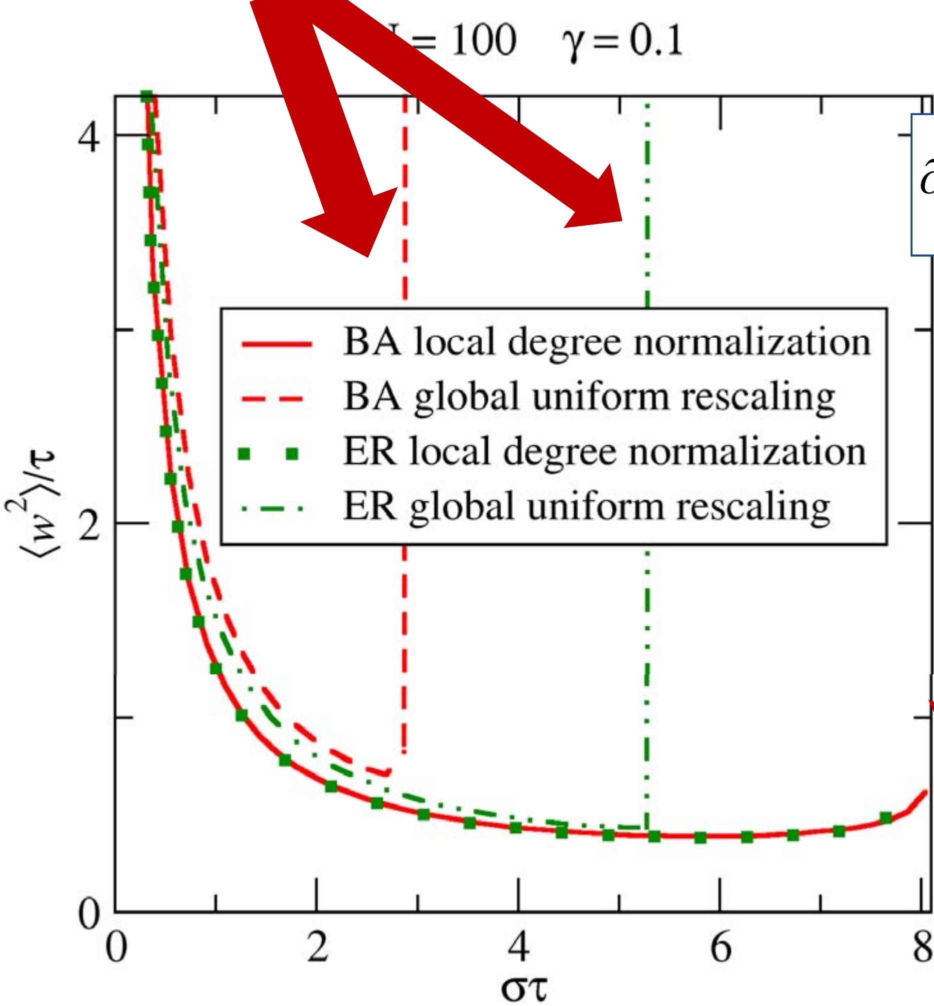
$\tau_o = \gamma\tau$

$$\partial_t h_i(t) = -\frac{\sigma}{\langle k \rangle}\sum_j A_{ij}[h_i(t-\gamma\tau) - h_j(t-\tau)] + \eta_i(t)$$

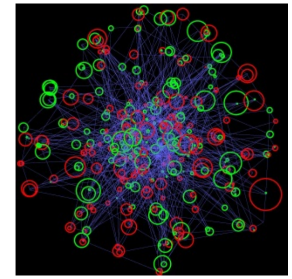
identical coupling cost:

$$\sum_{i,j}\frac{\sigma}{\langle k \rangle}A_{ij} = \sum_{i,j}\frac{\sigma}{k_i}A_{ij} = \sigma N$$

$$\partial_t h_i(t) = -\frac{\sigma}{k_i}\sum_j A_{ij}[h_i(t-\gamma\tau) - h_j(t-\tau)] + \eta_i(t)$$

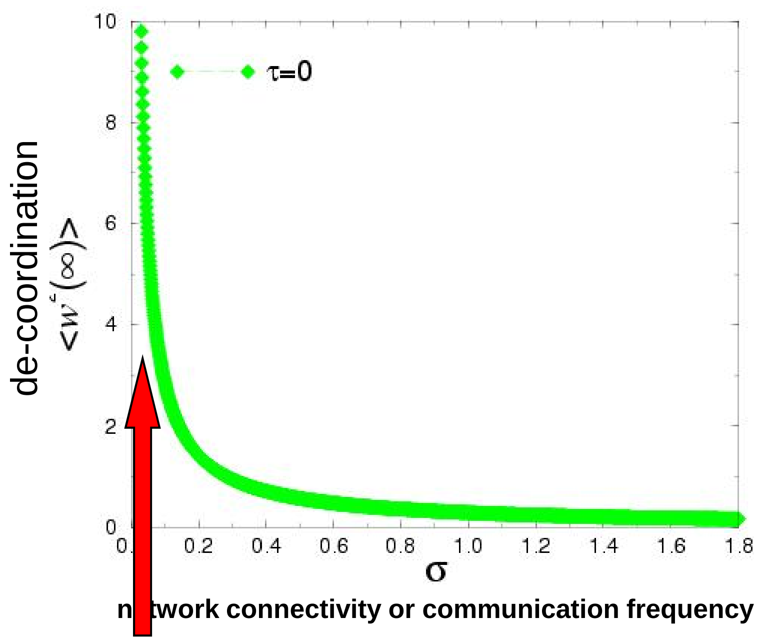


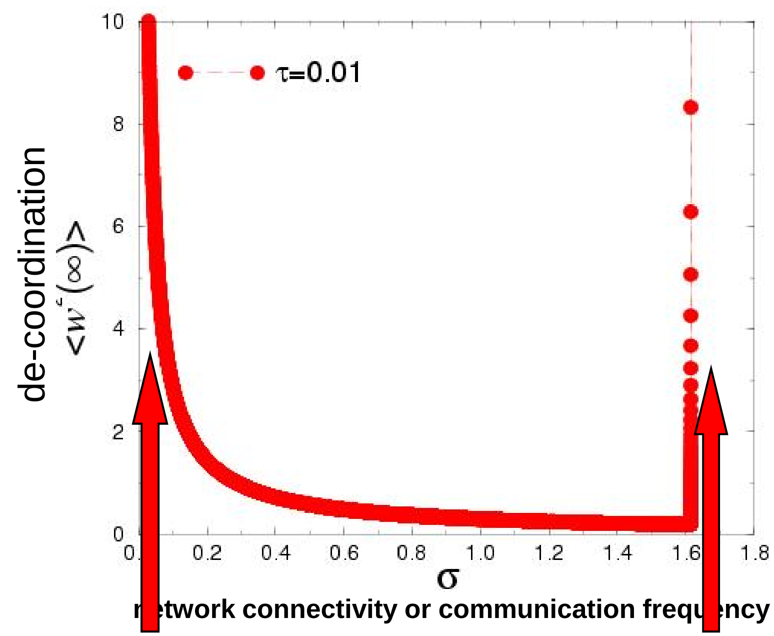
The Impact of Time Delays in Information and Communication Networks



- nodes/individuals constantly react to endogenous and exogenous information: coordination/agreement/consensus/alignment

- they react to the information or signal received from their neighbors possibly with some time lag τ (as result of finite transmission, decision, or execution delays)





low connectivity / no communication

low connectivity / no communication

high connectivity / "too much communication"

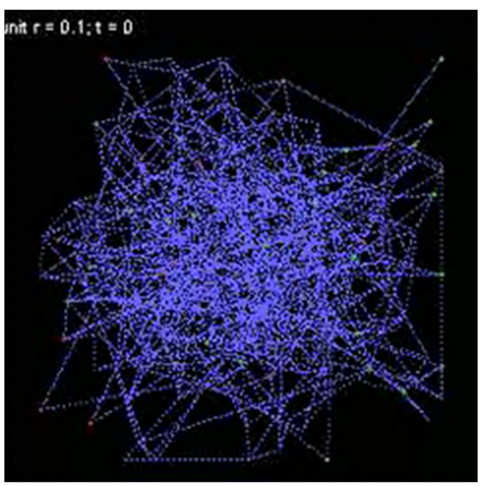
Hunt et al., *PRL* **105**, 068701 (2010)

Summary

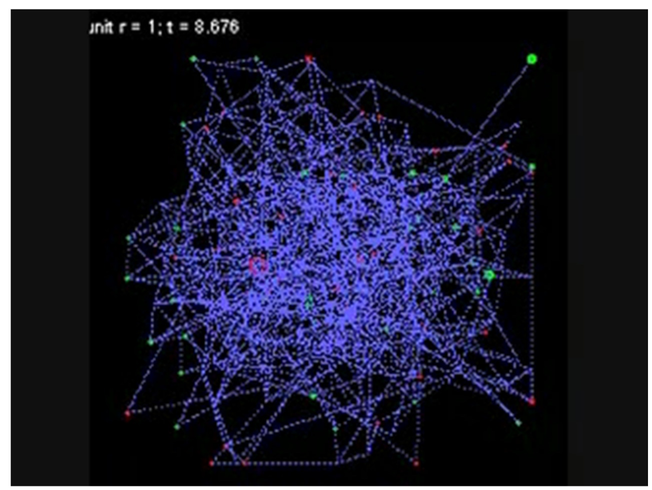
- Delays can destroy synchronization/coordination in networks
- Networks with large hubs can be particularly vulnerable in this regard
- Too much communication can cause more harm than good
- On the other hand, understanding the fundamental scaling properties of the underlying fluctuations (in particular the ones associated with the largest-eigenvalue mode) can guide optimization and trade-offs to control and to reduce these large fluctuations

D. Hunt, B.K. Szymanski, and G. Korniss, *Phys. Rev. Lett.* **105,** 068701 (2010).
D. Hunt, B.K. Szymanski, and G. Korniss, *Phys. Lett. A* **375**, 880 (2011).
D. Hunt, B.K. Szymanski, and G. Korniss, *Phys. Rev. E* **86**, 056114 (2012).

$\tau = 0$



$\tau > \tau_c$



Supported by DTRA, ARL NS-CTA, NSF