Polymers, Percolation and Logarithmic Minimal Models

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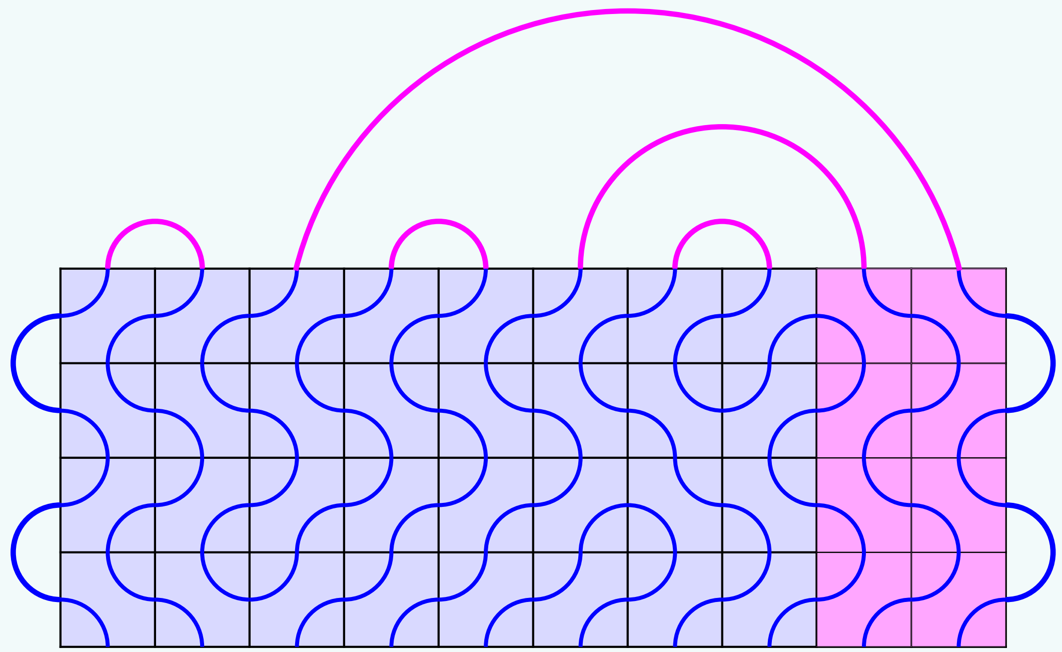

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Outline

- The Ising model of a magnet as a rational CFT with a finite operator content.
- Yang-Baxter integrability and Corner Transfer Matrices (CTMs).
- RSOS lattice models and minimal models $\mathcal{M}(m,m')$ as simplest rational CFTs.
- RSOS Generalized Order Parameters (GOPs) and their associated critical exponents and conformal weights.
- Logarithmic minimal models $\mathcal{LM}(p,p')$ as non-rational CFTs with an infinite number of scaling operators.
- Dense polymers and percolation as simplest examples of logarithmic theories.
- The *logarithmic limit* $\displaystyle\lim_{m,m'\to\infty,\ \frac{m}{m'}\to\frac{p}{p'}} \mathcal{M}(m,m') = \mathcal{LM}(p,p')$
- Off-critical solution of the logarithmic minimal models. Logarithmic limit of RSOS GOPs, associated critical exponents and conformal weights.

References:

- J.Rasmussen, *Logarithmic limits of minimal models*, Nucl. Phys. **B701** (2004) 516–528; *Jordan cells in logarithmic limits of conformal field theory*, Int. J. Mod. Phys. **A22** (2007) 67–82.
- PAP, J.Rasmussen, J.-B.Zuber, *Logarithmic minimal models*, J.Stat.Mech. P11017 (2006)
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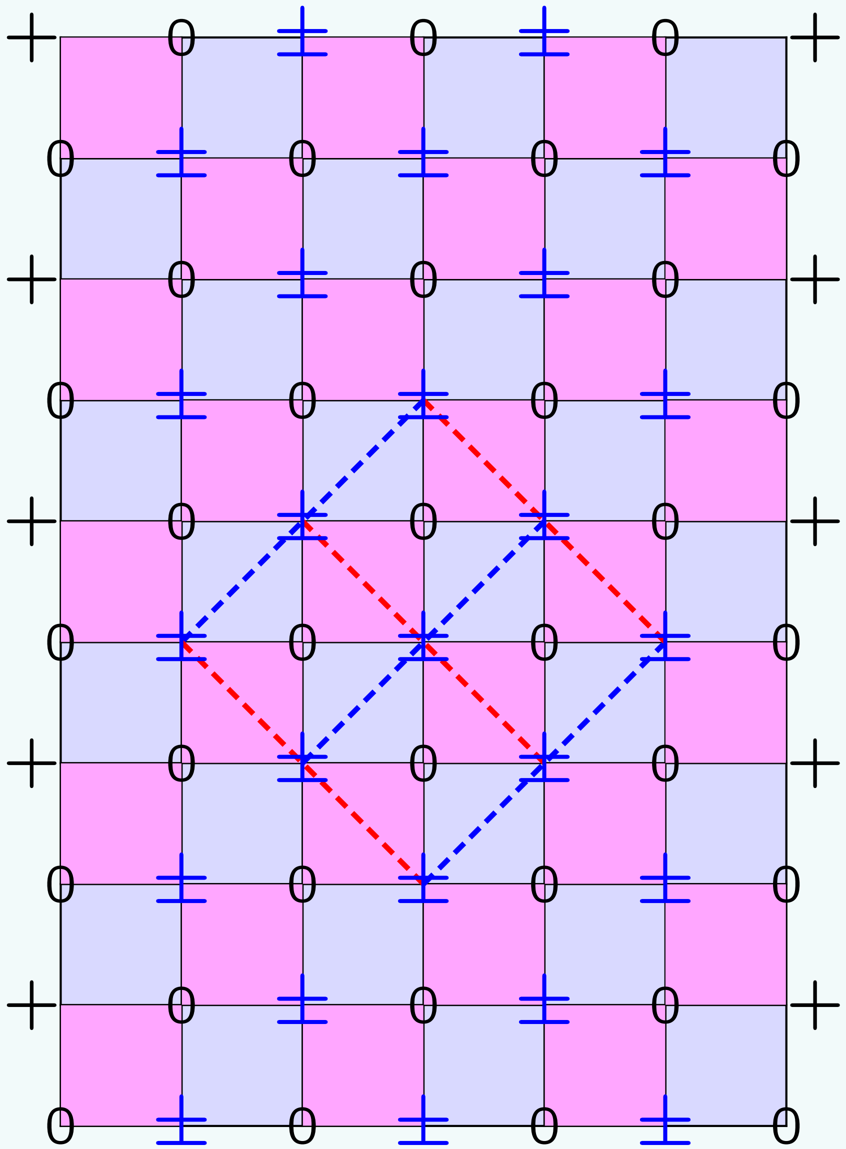
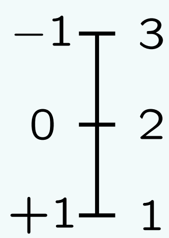
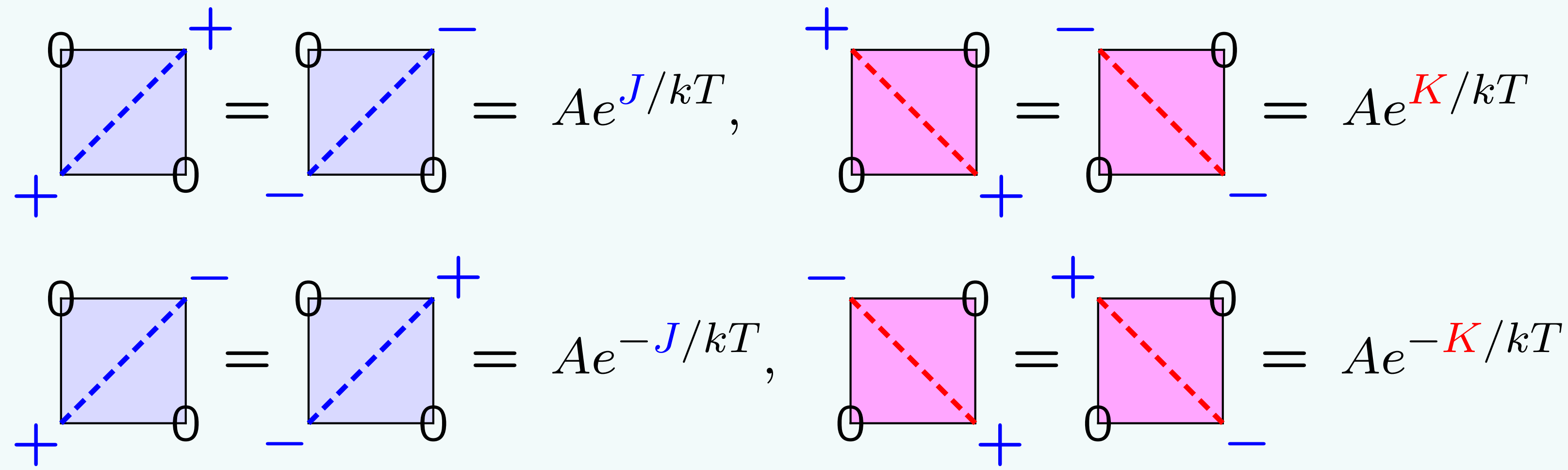
Square Lattice Ising Model



Spins: $\sigma_j = \pm 1$
$0 =$ frozen

$a \in A_3 =$ 

- Statistical Weights: $J, K =$ interaction strengths



- The weights can be parametrized as

$$W\begin{pmatrix} d & c \\ a & b \end{pmatrix}\!\Big| u\Big) = \frac{s(\lambda - u)}{s(\lambda)}\,\delta(a,c) + \frac{s(u)}{s(\lambda)}\sqrt{\frac{s(a\lambda)s(c\lambda)}{s(b\lambda)s(d\lambda)}}\,\delta(b,d)$$

heights: $a, b, c, d = 1, 2, 3;$ spins: $\sigma = 2 - a = 0, \pm 1$

$$\lambda = \frac{\pi}{4}, \qquad s(u) = \vartheta_1(u,t), \qquad 0 \le u \le \lambda, \quad 0 \le t \le 1$$

$$t^2 \sim T - T_c, \qquad \sinh\frac{2J}{kT_c}\sinh\frac{2K}{kT_c} = 1$$

$T =$ temperature, $u =$ spatial anisotropy $\sim \dfrac{J}{K}$

The partition function is $Z_N = \sum_{\text{spins}} \prod_{\text{faces}} W\begin{pmatrix} d & c \\ a & b \end{pmatrix}\!\Big| u\Big)$

Exact Solution of the Ising Model

- The Ising model was solved exactly in 1944 by Onsager for the limiting free energy f

$$-\frac{f}{kT} = \lim_{N\to\infty} \frac{1}{N} \log Z_N, \qquad N = \#\,\text{spins}$$

The specific heat $f''(T)$ diverges logarithmically at $T = T_c$ with a *critical exponent* $\alpha = 0$

$$f(T) \sim (T - T_c)^{2-\alpha}, \qquad T - T_c \to 0, \qquad f''(T) \sim \log(T - T_c), \qquad \alpha = 0_{\log}$$

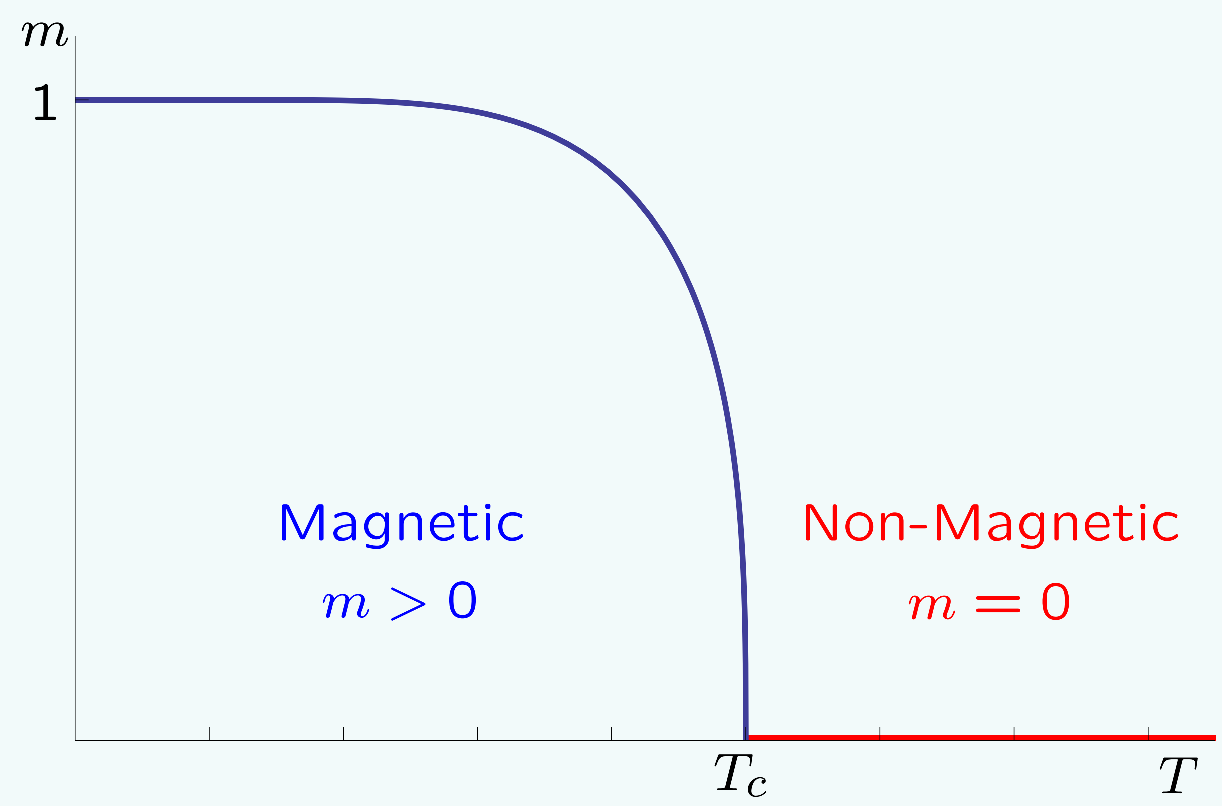
- The magnetization of the Ising model was calculated exactly in 1949 by C.N. Yang

$$m = \langle\sigma\rangle_+ = \lim_{N\to\infty} \frac{\displaystyle\sum_{\text{spins}} \prod_{\text{faces}} \sigma\, W\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right)}{\displaystyle\sum_{\text{spins}} \prod_{\text{faces}} W\!\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right)}$$

The magnetization vanishes above T_c with spontaneous magnetization below T_c

$$m \sim (T_c - T)^{\beta}, \qquad T - T_c \to 0-, \qquad \beta = 1/8$$

This One Point Function (OPF) is an example of an *order parameter*.

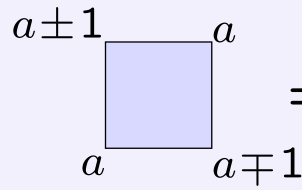


- Critical exponents such as α, β are *universal* (independent of the anisotropy or lattice structure) and described by a Conformal Field Theory (CFT) in the continuum scaling limit.

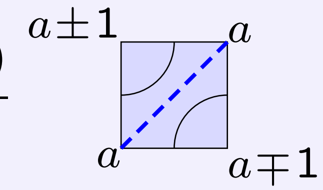
RSOS Models

- The statistical weights of the Restricted Solid-On-Solid (RSOS) models (Andrews-Baxter-Forrester, Forrester-Baxter 1984) are

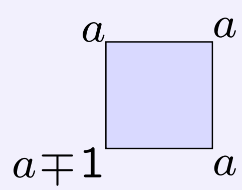
$$W\begin{pmatrix} a\pm 1 & a \\ a & a\mp 1 \end{pmatrix}\!\Big| u\Big) = \frac{s(\lambda-u)}{s(\lambda)}$$



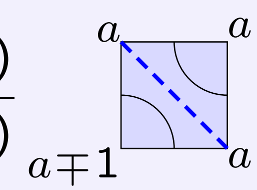
$$= \frac{s(\lambda-u)}{s(\lambda)}$$



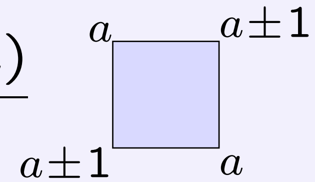
$$W\begin{pmatrix} a & a\pm 1 \\ a\mp 1 & a \end{pmatrix}\!\Big| u\Big) = \frac{s((a\pm 1)\lambda)}{s(a\lambda)}\frac{s(u)}{s(\lambda)}$$



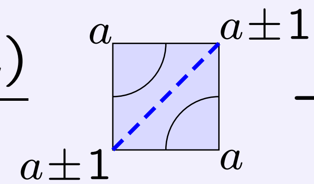
$$= \frac{s((a\pm 1)\lambda)}{s(a\lambda)}\frac{s(u)}{s(\lambda)}$$



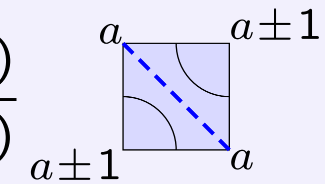
$$W\begin{pmatrix} a & a\pm 1 \\ a\pm 1 & a \end{pmatrix}\!\Big| u\Big) = \frac{s(a\lambda \pm u)}{s(a\lambda)}$$



$$= \frac{c(0)c((a\pm 1)\lambda \pm u)}{c((a\pm 1)\lambda)c(u)}\frac{s(\lambda-u)}{s(\lambda)}$$



$$+ \frac{c(\lambda)c(a\lambda\pm u)}{c((a\pm 1)\lambda)c(u)}\frac{s((a\pm 1)\lambda)}{s(a\lambda)}\frac{s(u)}{s(\lambda)}$$



Here $s(u) = \vartheta_1(u,t)$, $c(u) = \vartheta_4(u,t)$ are elliptic theta functions. At criticality, $t = 0$, $s(u) \mapsto \sin u$, $c(u) \mapsto 1$. The decomposition into decorated tiles allows to describe the nonlocal statistics of height clusters and loop connectivities (percolation properties).

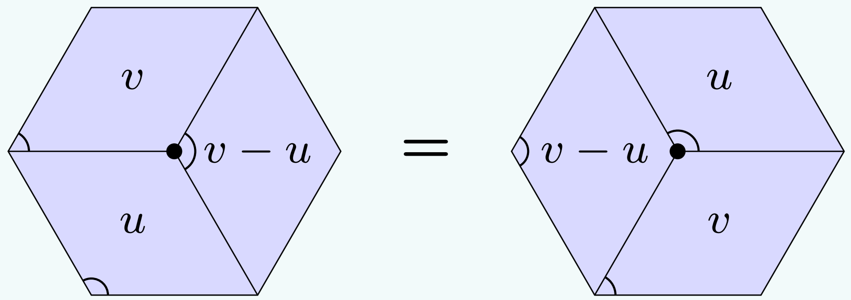
- The model dependent *crossing parameter* λ is

$$\lambda = \frac{(m'-m)\pi}{m'}, \qquad 2 \le m < m', \qquad m, m' \text{ coprime}, \qquad a = 1, 2, \ldots, m'-1$$

- In the continuum scaling limit, the critical RSOS models realize the minimal models $\mathcal{M}(m,m')$ (Belavin-Polyakov-Zamolodchikov 1984) — the simplest rational CFTs. Ising is $\mathcal{M}(3,4)$.

Yang-Baxter Integrability

- A 2-d lattice model is exactly solvable if the face weights satisfy the Yang-Baxter equation



At centre spin:

$$(\lambda - u) + v + (\lambda - v + u) = 2\lambda$$

$$\vartheta_1 + \vartheta_2 + \vartheta_3 = 2\pi$$

The RSOS models satisfy YBE and are integrable. The interactions depend on the *spatial anisotropy* u. The geometry of a face (rhombus) is fixed by the *anisotropy angle*

$$\vartheta = \frac{\pi(\lambda - u)}{\lambda} = \text{angle in marked corner}, \qquad \lambda = \text{crossing parameter}$$

- YBE implies commuting row and Corner Transfer Matrices (CTMs) and hence integrability.

- OPFs are calculated using Baxter's CTMs

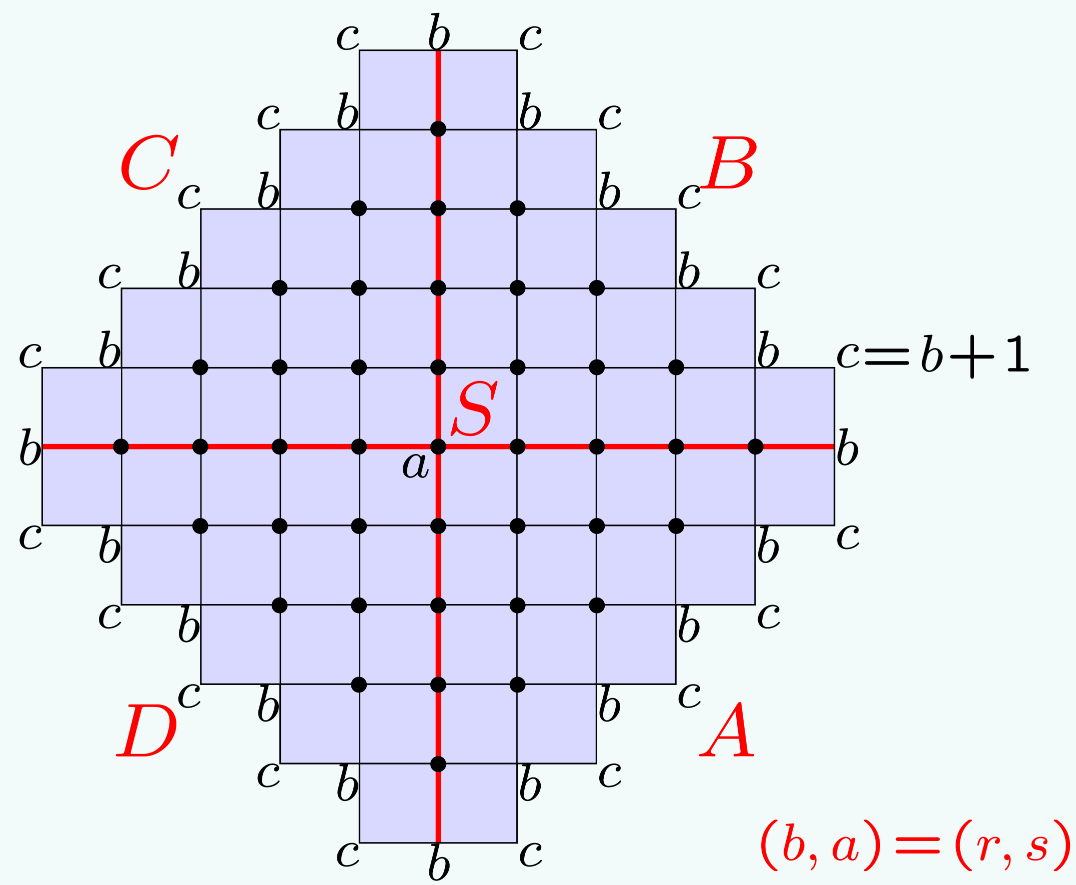
$$P_{r,s} = \langle \delta(a,s) \rangle_r = \lim_{N\to\infty} \frac{\mathrm{Tr}\, SABCD}{\mathrm{Tr} ABCD}$$

where S fixes the center height a to the height s and r labels the boundary conditions with heights $b = r$ and $c = r+1$.

- For the Ising model, the magnetization is

$$m = P_{1,1} - P_{1,3} = \langle \delta(a,1) \rangle_1 - \langle \delta(a,3) \rangle_1$$

with spins $b, c \approx +, 0$ on the boundary.



Ising Model Operator Content

- The Ising CFT $\mathcal{M}(3,4)$ is characterized by a *central charge* $c = \frac{1}{2}$. It admits three operators $\{I, \sigma, \varepsilon\}$ with associated *conformal dimensions* Δ and conjugate boundary conditions:

operator	I	σ	ε	$I =$ identity	$\sigma =$ magnetization	$\varepsilon =$ energy
bc	$+$	Free	$-$	$\Delta_I = 0$	$\Delta_\sigma = \frac{1}{16}$	$\Delta_\varepsilon = \frac{1}{2}$

- The identity conformal dimension is $\Delta_I = \Delta_{1,1} \equiv \Delta_{2,3} = 0$.

Other conformal dimensions are determined by critical exponent scaling relations

$$\psi = \text{free energy} \sim (T - T_c)^{2-\alpha}, \qquad \alpha = 0 \;\Rightarrow\; \Delta_\varepsilon = \frac{1-\alpha}{2-\alpha} = \frac{1}{2} = \Delta_{1,3} \equiv \Delta_{2,1}$$

$$m = \text{magnetization} \sim (T - T_c)^{\beta}, \qquad \beta = \frac{1}{8} \;\Rightarrow\; \Delta_\sigma = \frac{1}{2}\beta = \frac{1}{16} = \Delta_{1,2} \equiv \Delta_{2,2}$$

- The conformal dimensions $\Delta = \Delta_{r,s}$ and operator content are neatly encoded in a 2×3 Kac table with the symmetry $\Delta_{r,s} = \Delta_{m-r,m'-s}$. The Kac labels (r,s) coincide with the CTM boundary condition labels.

s		
3	$\frac{1}{2}$	0
2	$\frac{1}{16}$	$\frac{1}{16}$
1	0	$\frac{1}{2}$
	1	2 $\quad r$

- The conformal data consists of the central charge $c = \frac{1}{2}$, the conformal dimensions $\Delta_{r,s}$ and the characters (generating functions of the conformal spectra)

$$\text{ch}_{1,1}(q) = \text{ch}_0(q), \quad \text{ch}_{1,2}(q) = \text{ch}_{\frac{1}{16}}(q), \quad \text{ch}_{2,1}(q) = \text{ch}_{\frac{1}{2}}(q)$$

Minimal Model Operator Content

- In the continuum scaling limit, the critical RSOS models realize the minimal models — the simplest rational CFTs with a finite number of scaling operators. The conformal data is

$$c = 1 - \frac{6(m-m')^2}{mm'}$$

$$\Delta^{m,m'}_{r,s} = \frac{(rm'-sm)^2-(m-m')^2}{4mm'}$$

$$\mathrm{ch}^{m,m'}_{r,s}(q) = \frac{q^{-\frac{c}{24}+\Delta^{m,m'}_{r,s}}}{\prod_{n=1}^{\infty}(1-q^n)} \sum_{k=-\infty}^{\infty}\left[q^{k(kmm'+rm'-sm)} - q^{(km+r)(km'+s)}\right]$$

- Kac tables of conformal weights $\Delta^{m,m'}_{r,s}$ for $\mathcal{M}(3,4)$, $\mathcal{M}(2,5)$, $\mathcal{M}(4,7)$, $\mathcal{M}(5,7)$. The pink boxes are associated with order parameters.

$\mathcal{M}(3,4)$:

s \ r	1	2
3	$\frac{1}{2}$	0
2	$\frac{1}{16}$	$\frac{1}{16}$
1	0	$\frac{1}{2}$

$\mathcal{M}(2,5)$:

s \ r	1
4	0
3	$-\frac{1}{5}$
2	$-\frac{1}{5}$
1	0

$\mathcal{M}(4,7)$:

s \ r	1	2	3
6	$\frac{5}{2}$	$\frac{13}{16}$	0
5	$\frac{10}{7}$	$\frac{27}{112}$	$-\frac{1}{14}$
4	$\frac{9}{14}$	$-\frac{5}{112}$	$\frac{1}{7}$
3	$\frac{1}{7}$	$-\frac{5}{112}$	$\frac{9}{14}$
2	$-\frac{1}{14}$	$\frac{27}{112}$	$\frac{10}{7}$
1	0	$\frac{13}{16}$	$\frac{5}{2}$

$\mathcal{M}(5,7)$:

s \ r	1	2	3	4
6	$\frac{15}{4}$	$\frac{9}{5}$	$\frac{11}{20}$	0
5	$\frac{16}{7}$	$\frac{117}{140}$	$\frac{3}{35}$	$\frac{1}{28}$
4	$\frac{33}{28}$	$\frac{8}{35}$	$-\frac{3}{140}$	$\frac{3}{7}$
3	$\frac{3}{7}$	$-\frac{3}{140}$	$\frac{8}{35}$	$\frac{33}{28}$
2	$\frac{1}{28}$	$\frac{3}{35}$	$\frac{117}{140}$	$\frac{16}{7}$
1	0	$\frac{11}{20}$	$\frac{9}{5}$	$\frac{15}{4}$

One Point Functions of RSOS Models

- The RSOS OPFs were calculated by Forrester-Baxter using CTMs

$$P_{r,s} = \langle \delta(a,s) \rangle_b = \lim_{N\to\infty} \frac{\mathrm{Tr}\, SABCD}{\mathrm{Tr} ABCD}$$

where S fixes the center height a to the value s. The $m-1$ groundstates are labelled by $b = b(r) = \lfloor \frac{rm'}{m} \rfloor$ with $r = 1, 2, \dots, m-1$. Explicitly,

$$\begin{aligned} P_{r,s} &= \frac{q^{\frac{c}{24} - \Delta^{m,m'}_{r,s} + \frac{(s-r)(s-r-1)}{4}} E(q^{\frac{s}{2}}, q^{\frac{m'}{2(m'-m)}})(q)_\infty}{E(-q^{\frac{1}{2}}, q^2) E(q^{\frac{r}{2}}, q^{\frac{m}{2(m'-m)}})} \, \mathrm{ch}^{m,m'}_{r,s}(q) \\ &= \sqrt{\frac{2m}{m'}} \frac{\eta(t^{\frac{m'}{m'-m}})\, \vartheta_1(\frac{s\pi(m'-m)}{m'}, t)}{\vartheta_4(0, t^{\frac{m'}{m'-m}})\, \vartheta_1(\frac{\pi r(m'-m)}{m}, t^{\frac{m'}{m}})} \sum_{(r's') \in \mathcal{J}} \mathcal{S}_{rs;r's'} \, \mathrm{ch}^{m,m'}_{r',s'}(t^{\frac{m'}{m'-m}}) \end{aligned}$$

- The low-temperature nome is $q = e^{-4\pi\lambda/\epsilon}$ and the critical nome is $t = e^{-\epsilon}$

$$E(x,q) = \prod_{n=1}^{\infty} (1 - q^{n-1}x)(1 - q^n x^{-1})(1 - q^n), \quad \eta(q) = q^{1/24}(q)_\infty, \quad (q)_\infty = \prod_{n=1}^{\infty}(1 - q^n)$$

and the modular matrix $\mathcal{S}$ is

$$\mathcal{S}_{rs;r's'} = \sqrt{\frac{8}{mm'}} (-1)^{(r'+s')(r+s)} \sin \frac{\pi(m'-m)rr'}{m} \sin \frac{\pi(m'-m)ss'}{m'}$$

- The $\frac{1}{2}(m-1)(m'-1)$ independent OPFs satisfy $P_{m-r,m'-s} = P_{r,s}$ so we restrict (r,s) to the bottom-left $\mathcal{J}$ of the Kac table.

Critical Exponents of Generalized Order Parameters

- For $m'-m>1$, some face weights are negative and all OPFs diverge at criticality $t\to 0$

$$P_{r,s}\sim (t^{\frac{\pi}{\lambda}})^{\Delta^{m,m'}_{r_0,s_0}},\qquad \Delta^{m,m'}_{r_0,s_0}=\frac{1-(m'-m)^2}{4mm'}=\underset{(r,s)\in\mathcal{J}}{\text{Min}}\,\Delta^{m,m'}_{r,s}<0,\qquad m'-m>1$$

Forrester-Baxter remark that, it is not possible to define order parameters in the usual sense.

- Generalizing Huse 1984, we introduce Generalized Order Parameters (GOPs)

$$R_{r'',s''}=\sum_{(r,s)\in\mathcal{J}}\mathcal{S}_{r''s'';rs}\,\frac{\sin\frac{\pi r(m'-m)}{m}}{\sin\frac{\pi s(m'-m)}{m'}}\,P_{r,s}$$

Since $\mathcal{S}^2=I$, the modular matrix $\mathcal{S}$ effectively undoes the modular $\mathcal{S}$ matrix introduced by the conjugate modulus transformation.

- Defining new observables $\mathcal{O}_{r,s}$ as ratios of the GOPs yields

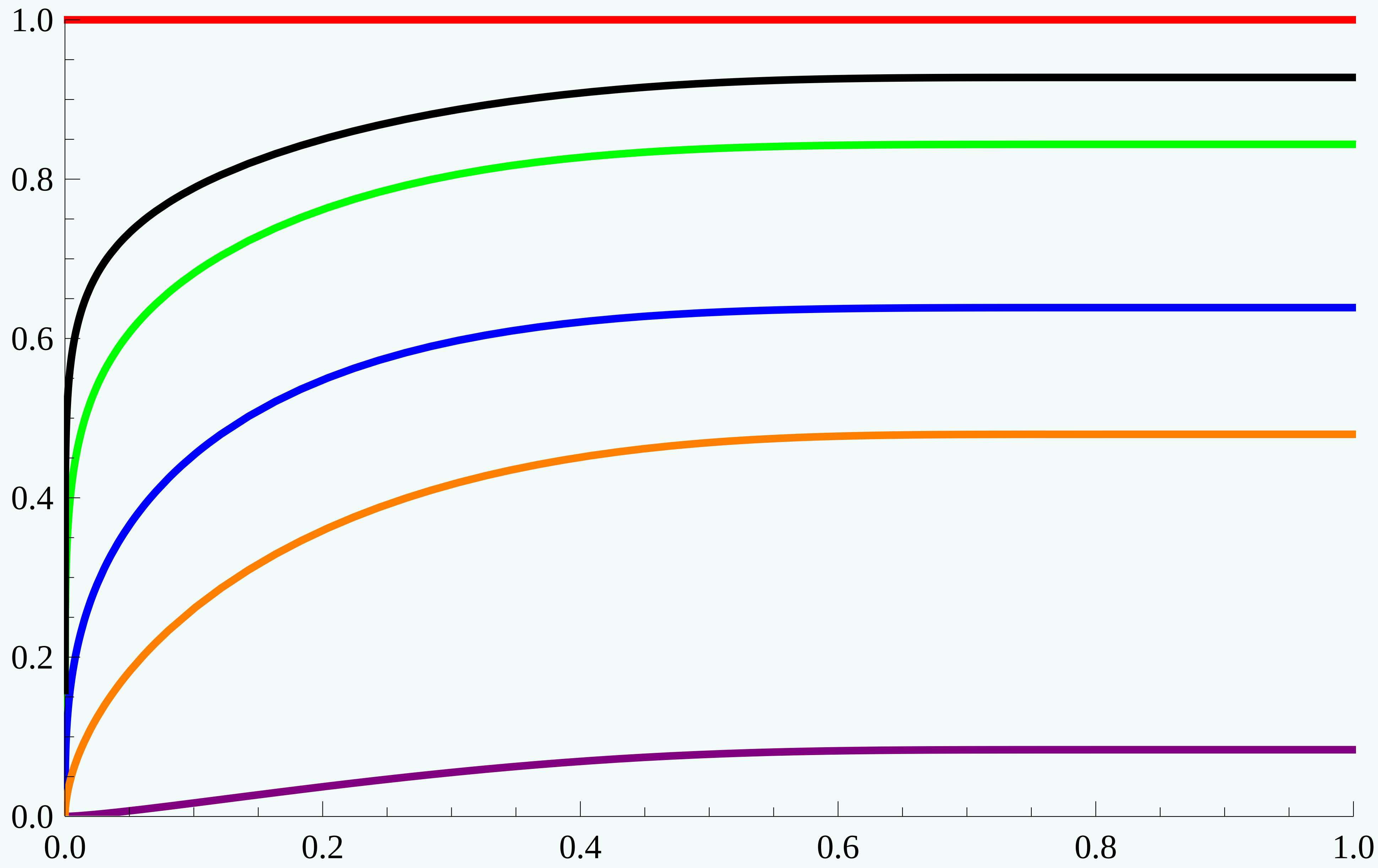
$$\mathcal{O}_{r,s}=\frac{R_{r,s}}{R_{1,1}}\sim (t^{\frac{\pi}{\lambda}})^{\Delta^{m,m'}_{r,s}}+(t^{\frac{\pi}{\lambda}})^{\Delta^{m,m'}_{r_0,s_0}}O(t^2),\qquad \mathcal{O}_{r,s}\sim (t^2)^{\beta_{r,s}}$$

where t^2 measures the departure from criticality. The second term is of smaller order if (r,s) satisfies $(m'r-ms)^2<1+8m(m'-m)$ yielding the critical exponents

$$\beta_{r,s}=(2-\alpha)\Delta^{m,m'}_{r,s}=\frac{(rm'-sm)^2-(m'-m)^2}{8m(m'-m)},\qquad 2-\alpha=\frac{\pi}{2\lambda}=\frac{m'}{2(m'-m)}$$

Here α comes from the behaviour of the known free energy.

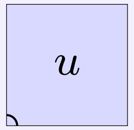
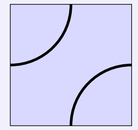
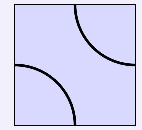
Plots of Order Parameters for $\mathcal{M}(4,7)$



- Plot of the observables $\mathcal{O}_{r,s}$, as a function of t, for the minimal model $\mathcal{M}(4,7)$. From top to bottom, we plot $\mathcal{O}_{1,1}, \frac{1}{\mathcal{O}_{2,3}}, \frac{1}{\mathcal{O}_{1,2}}, \mathcal{O}_{1,3}, \mathcal{O}_{2,2}, \mathcal{O}_{1,4}$ corresponding to $|\Delta_{r,s}| = 0, \frac{5}{112}, \frac{1}{14}, \frac{1}{7}, \frac{27}{112}, \frac{9}{14}$ in increasing order with critical exponents $\beta_{r,s} = \frac{7}{6}\Delta_{r,s}$. As expected for order parameters, these observables are nonnegative, vanish at criticality and are increasing functions of t.

Critical Logarithmic Minimal Models $\mathcal{LM}(p,p')$

- The face operators are defined in the planar Temperley-Lieb algebra (Jones 1999) by

$X(u) =$  $= \dfrac{\sin(\lambda-u)}{\sin\lambda}$  $+ \dfrac{\sin u}{\sin\lambda}$  ;

$$X_j(u) = \frac{\sin(\lambda-u)}{\sin\lambda}\, I + \frac{\sin u}{\sin\lambda}\, e_j$$

$1 \le p < p'$ coprime integers, $\quad \lambda = \dfrac{(p'-p)\pi}{p'} =$ crossing parameter

$u =$ spectral parameter, $\quad \beta = 2\cos\lambda =$ fugacity of loops

$$Z_N = \sum_{\text{loop configs}} \left[\frac{\sin(\lambda-u)}{\sin\lambda}\right]^{N_1} \left[\frac{\sin u}{\sin\lambda}\right]^{N_2} \beta^{\#\text{ loops}}$$

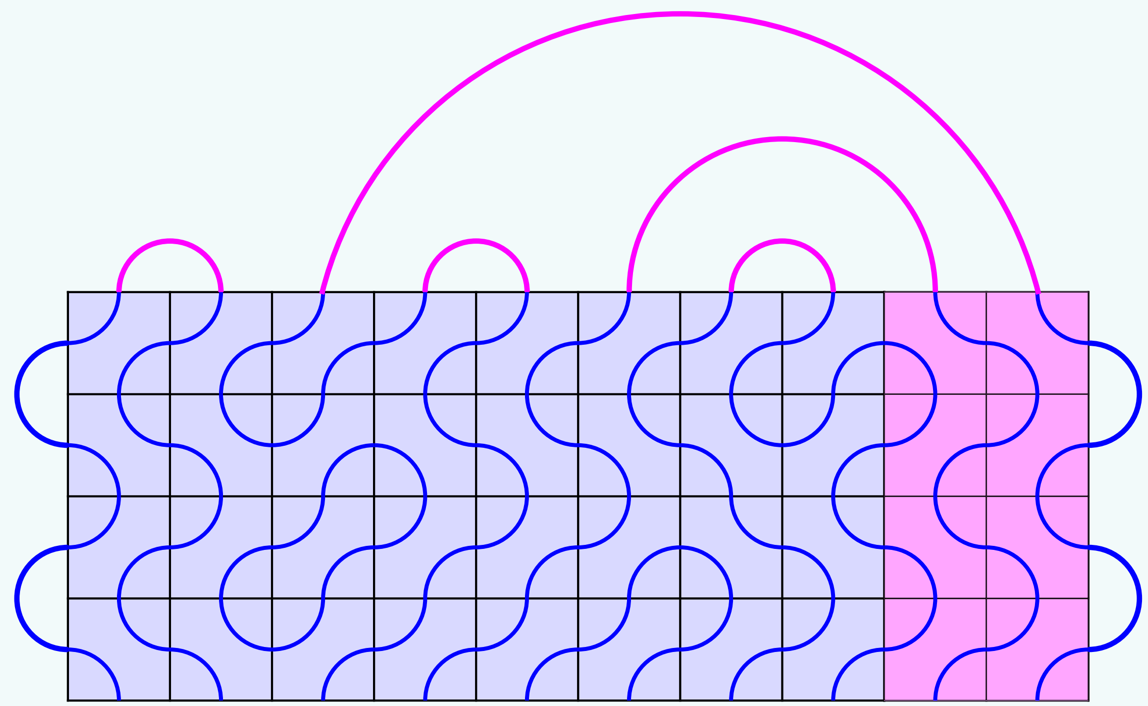
Planar Algebra
(Temperley-Lieb Algebra)

↓ YBE

Nonlocal Statistical Mechanics
(Yang-Baxter Integrable Link Models)

↓ continuum limit ↑ lattice realization

Logarithmic CFTs
(Logarithmic Minimal Models)



- Transfer matrix acts on space of link states
- No local degrees of freedom
- Only nonlocal degrees of freedom

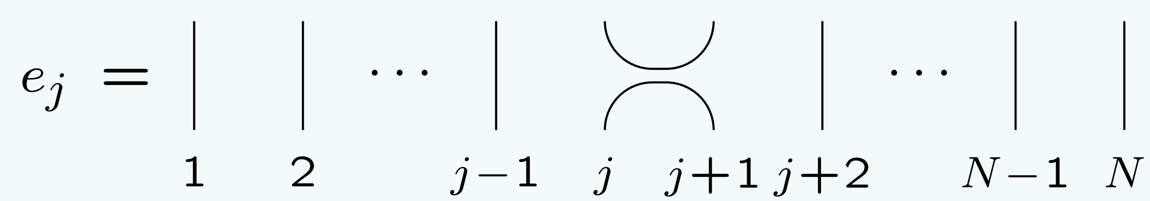
Temperley-Lieb Algebra

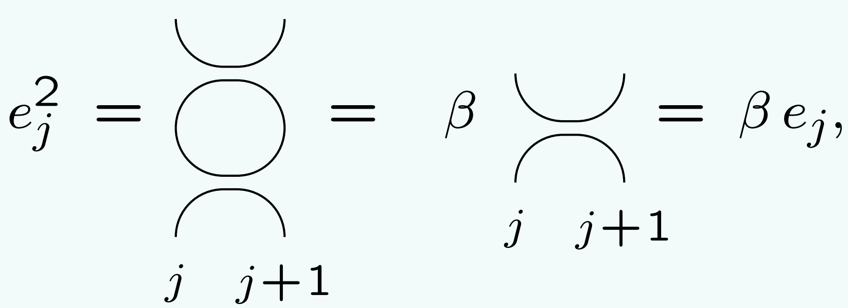
$$\begin{cases} \text{Critical RSOS (minimal) models} = \text{height representation of TL} \\ \text{Critical logarithmic minimal models} = \text{loop representation of TL} \end{cases}$$

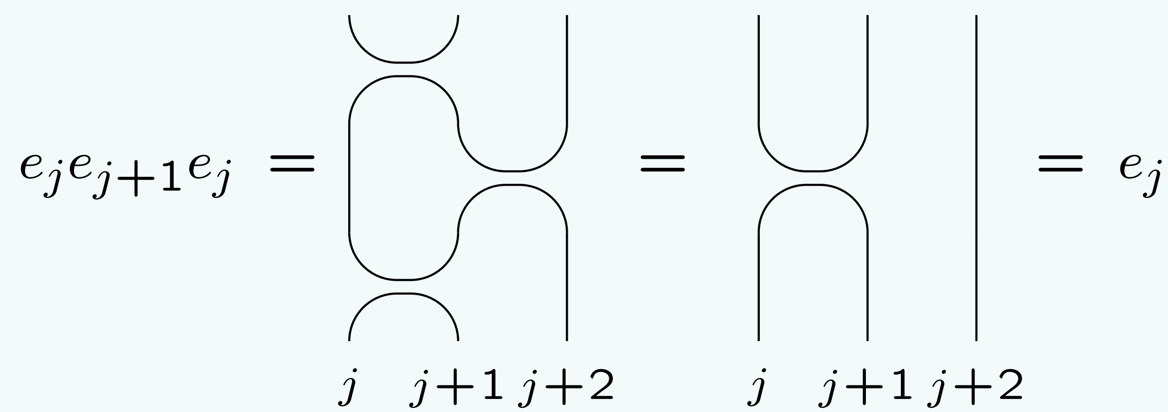
- The linear TL algebra (Temperley-Lieb 1971) is generated by $e_1, \ldots, e_{N-1}$ and the identity I acting on N strings

$$\begin{cases} e_j^2 = \beta\, e_j, & \\ e_j\, e_k\, e_j = e_j, \quad |j-k| = 1, & j,k = 1,2,\ldots,N-1; \quad \beta = 2\cos\lambda \\ e_j\, e_k = e_k\, e_j, \quad |j-k| > 1 & \end{cases}$$

- In the loop representation, the TL generators e_j are given graphically by *monoids*



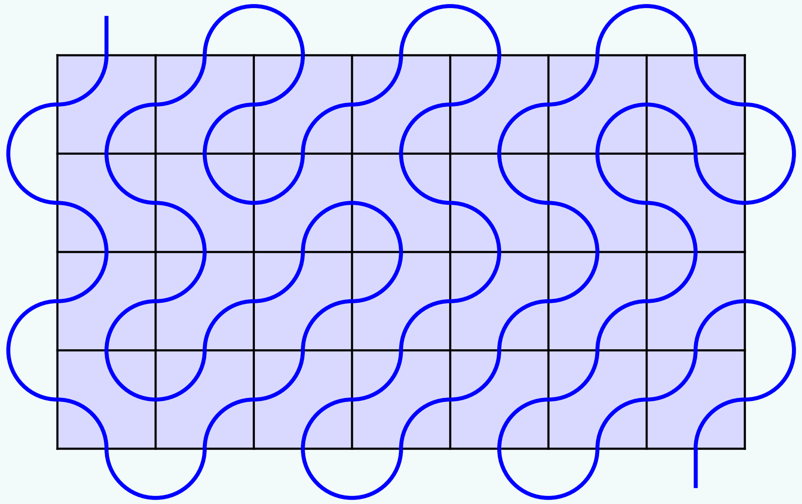




- Following Jones (1999), the TL algebra extends to a planar algebra where multiplication (in arbitrary directions) is implemented diagrammatically.

Polymers and Percolation on the Lattice

- **Critical Dense Polymers:** de Gennes 1972 $\quad (p,p') = (1,2), \qquad \lambda = \frac{\pi}{2}$

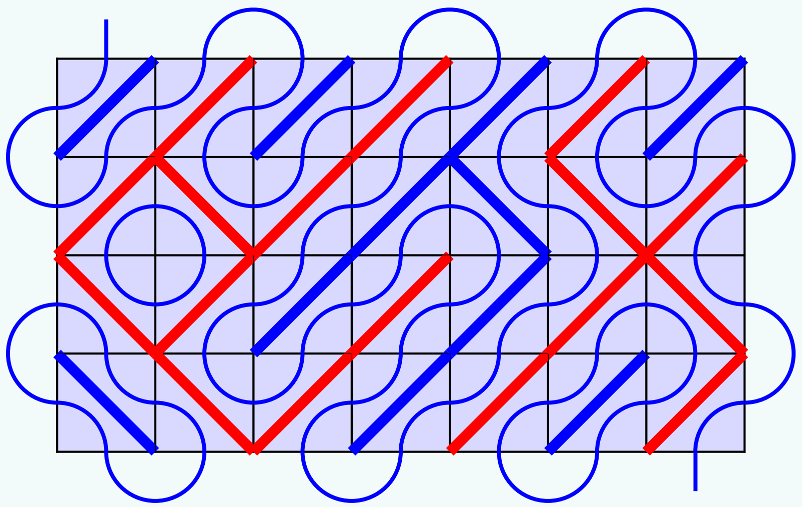


$$d_{path}^{SLE} = 2 - 2\Delta_{p,p'-1} = 2, \qquad \kappa = \frac{4p'}{p} = 8$$

$\Delta_{1,1} = 0$ lies outside rational $\mathcal{M}(1,2)$ Kac table

$\beta = 0 \;\Rightarrow\;$ no loops $\;\Rightarrow\;$ space filling dense polymer (Peano curve)

- **Critical Percolation:** Broadbent/Hammersley 1957 $\quad (p,p') = (2,3), \quad \lambda = \frac{\pi}{3} = 2u$



$$d_{path}^{SLE} = 2 - 2\Delta_{p,p'-1} = \frac{7}{4} < 2, \qquad \kappa = \frac{4p'}{p} = 6$$

$\Delta_{2,2} = \frac{1}{8}$ lies outside rational $\mathcal{M}(2,3)$ Kac table

$\beta = 1 \qquad \Rightarrow \qquad$ local stochastic process

Kesten 1980: Critical probability $= p_c = \sin(\lambda - u) = \sin u = \frac{1}{2}$

Duplantier 1988: Loop model $\Leftrightarrow$ Critical bond percolation on the blue square lattice

Critical Dense Polymer $\mathcal{LM}(1,2)$ Kac Table

- **Central charge:** $(p,p')=(1,2)$

$$c \;=\; 1-\frac{6(p-p')^2}{pp'} \;=\; -2$$

- **Infinitely extended Kac table of conformal weights:**

$$\begin{aligned}\Delta_{r,s} &= \frac{(p'r-ps)^2-(p-p')^2}{4pp'}\\ &= \frac{(2r-s)^2-1}{8}, \qquad r,s=1,2,3,\ldots\end{aligned}$$

- **Kac representation characters:**

$$\chi_{r,s}(q)=q^{-c/24}\,\frac{q^{\Delta_{r,s}}(1-q^{rs})}{\prod_{n=1}^{\infty}(1-q^n)}$$

- The conformal weights in pink shaded boxes are associated to order parameters.
- Note $\mathcal{M}(1,2)$ has an empty Kac table.

s							
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	$\frac{63}{8}$	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\cdots$
9	6	3	1	0	0	1	$\cdots$
8	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\cdots$
7	3	1	0	0	1	3	$\cdots$
6	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\cdots$
5	1	0	0	1	3	6	$\cdots$
4	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\cdots$
3	0	0	1	3	6	10	$\cdots$
2	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\frac{99}{8}$	$\cdots$
1	0	1	3	6	10	15	$\cdots$
	1	2	3	4	5	6	r

Critical Percolation $\mathcal{LM}(2,3)$ Kac Table

- **Central charge:** $(p,p') = (2,3)$

$$c \;=\; 1 - \frac{6(p-p')^2}{pp'} \;=\; 0$$

- **Infinitely extended Kac table of conformal weights:**

$$\Delta_{r,s} \;=\; \frac{(p'r-ps)^2-(p-p')^2}{4pp'} \;=\; \frac{(3r-2s)^2-1}{24}, \qquad r,s=1,2,3,\ldots$$

- **Kac representation characters:**

$$\chi_{r,s}(q) = q^{-c/24}\,\frac{q^{\Delta_{r,s}}(1-q^{rs})}{\prod_{n=1}^{\infty}(1-q^n)}$$

- The conformal weights in pink shaded boxes are associated to order parameters.

s							
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	12	$\frac{65}{8}$	5	$\frac{21}{8}$	1	$\frac{1}{8}$	$\cdots$
9	$\frac{28}{3}$	$\frac{143}{24}$	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\cdots$
8	7	$\frac{33}{8}$	2	$\frac{5}{8}$	0	$\frac{1}{8}$	$\cdots$
7	5	$\frac{21}{8}$	1	$\frac{1}{8}$	0	$\frac{5}{8}$	$\cdots$
6	$\frac{10}{3}$	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\cdots$
5	2	$\frac{5}{8}$	0	$\frac{1}{8}$	1	$\frac{21}{8}$	$\cdots$
4	1	$\frac{1}{8}$	0	$\frac{5}{8}$	2	$\frac{33}{8}$	$\cdots$
3	$\frac{1}{3}$	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$	$\frac{10}{3}$	$\frac{143}{24}$	$\cdots$
2	0	$\frac{1}{8}$	1	$\frac{21}{8}$	5	$\frac{65}{8}$	$\cdots$
1	0	$\frac{5}{8}$	2	$\frac{33}{8}$	7	$\frac{85}{8}$	$\cdots$
	1	2	3	4	5	6	r

Rational $\mathcal{M}(2,3)$ Kac table

Logarithmic Limit

- Symbolically, the "*logarithmic limit*" (Rasmussen 2004, 2007) of the minimal CFTs is

$$\lim_{m,m'\to\infty,\ \frac{m}{m'}\to\frac{p}{p'}} \mathcal{M}(m,m') = \mathcal{LM}(p,p'), \qquad 1 \le p < p', \quad p,p' \text{ coprime}$$

The limit is taken through coprime pairs (m,m') in the continuum scaling limit, after the thermodynamic limit $N\to\infty$.

- Nontrivial Jordan cells can emerge in this limit, but the equality means identification of the spectra of these CFTs:

$$c^{m,m'} = 1 - \frac{6(m-m')^2}{mm'} \;\to\; 1 - \frac{6(p-p')^2}{pp'} = c^{p,p'}$$

$$\Delta_{r,s}^{m,m'} = \frac{(rm'-sm)^2-(m-m')^2}{4mm'} \;\to\; \frac{(rp'-sp)^2-(p-p')^2}{4pp'} = \Delta_{r,s}^{p,p'}$$

$$\text{ch}_{r,s}^{m,m'}(q) = \frac{q^{-\frac{c}{24}+\Delta_{r,s}^{m,m'}}}{(q)_\infty}\sum_{k=-\infty}^{\infty}\Big[q^{k(kmm'+rm'-sm)}-q^{(km+r)(km'+s)}\Big] \to q^{-\frac{c}{24}+\Delta_{r,s}^{p,p'}}\,\frac{(1-q^{rs})}{(q)_\infty} = \chi_{r,s}^{p,p'}(q)$$

- The logarithmic limit can be applied to the minimal (RSOS) models off-criticality

$$\begin{array}{ccc} \mathcal{M}(m,m') & \xrightarrow{\ t\sim\varphi_{1,3}\ } & \mathcal{M}(m,m';t) \\ \Big\downarrow{\scriptstyle\text{log}} & & \Big\downarrow{\scriptstyle\text{log}} \\ \mathcal{LM}(p,p') & \xrightarrow{\ t\sim\varphi_{1,3}\ } & \mathcal{LM}(p,p';t) \end{array}$$

Logarithmic Limit of GOPs

- There is no simple conjugate modulus transformation on the infinity of logarithmic Kac characters $\chi^{p,p'}_{r,s}(q)$ since the entries of the $\mathcal{S}$ matrix have a common prefactor $\sqrt{\frac{8}{mm'}}$ which vanishes as $m, m' \to \infty$.

- Even so, the logarithmic limit of the observables $\mathcal{O}_{r,s}$ (in which the problematic prefactors cancel out in the ratio) are well defined and admit a Taylor expansion about $t = 0$

$$\mathcal{O}^{\infty}_{r,s} = \lim_{m,m'\to\infty,\ \frac{m}{m'}\to\frac{p}{p'}} \frac{R_{r,s}}{R_{1,1}} \sim (t^{\frac{\pi}{\lambda}})^{\Delta^{p,p'}_{r,s}} + (t^{\frac{\pi}{\lambda}})^{\Delta^{p,p'}_{r_0,s_0}} O(t^2), \qquad \lambda = \frac{(p'-p)\pi}{p'}$$

- We conclude that, corresponding to the perturbation off-criticality,

$$\mathcal{O}^{\infty}_{r,s} \sim (t^2)^{\beta_{r,s}}, \qquad \beta_{r,s} = (2-\alpha)\Delta^{p,p'}_{r,s} = \frac{(rp'-sp)^2 - (p'-p)^2}{8p(p'-p)}, \qquad 2-\alpha = \frac{\pi}{2\lambda} = \frac{p'}{2(p'-p)}$$

- This constructs limiting observables with associated critical exponents for all Kac conformal weights satisfying

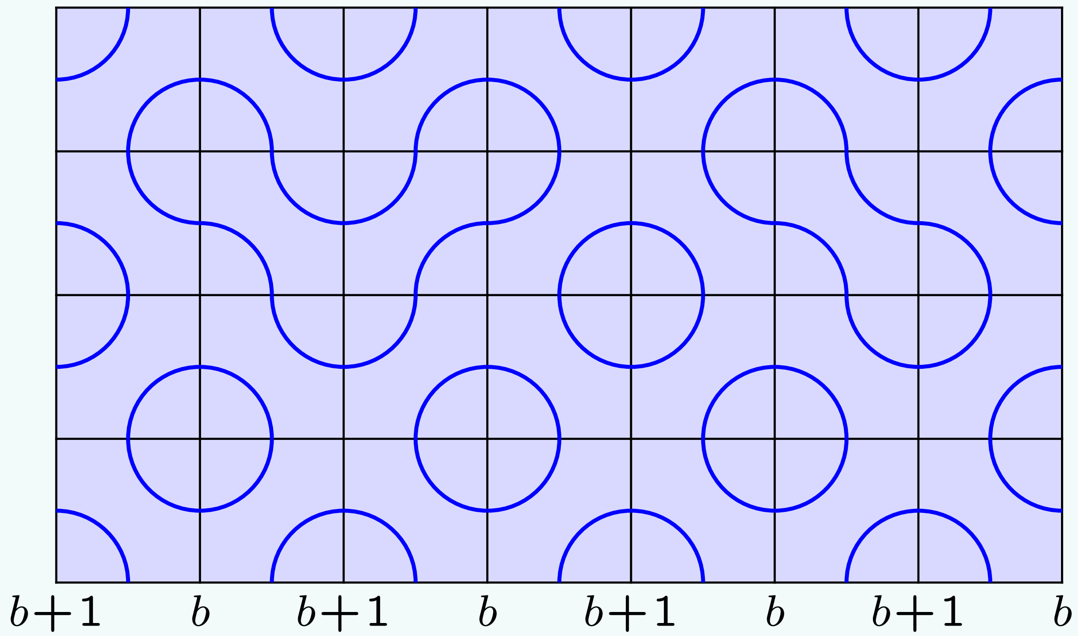
$$\boxed{\Delta^{p,p'}_{r,s} < \Delta^{p,p'}_{r_0,s_0} + \frac{2(p'-p)}{p'} = \frac{(p'-p)(9p-p')}{4pp'}}$$

These occur for (r, s) in the infinitely extended Kac tables satisfying

$$\boxed{(p'r-ps)^2 < 8p(p'-p)}$$

Off-Critical Dense Polymers

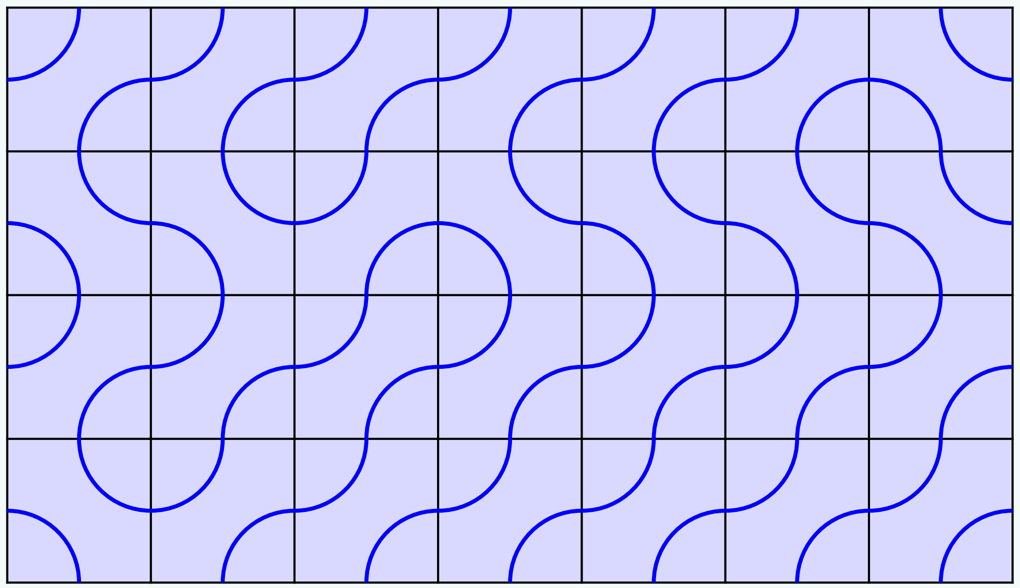
- Typical configurations with heights suppressed:



$T \to 0$ $(t \to 1)$

Small closed loops (no long chains)

Ordered heights (infinite # of flat groundstates)



$T \to T_c$ $(t \to 0)$

Long polymer chains (no closed loops)

Disordered RSOS heights

- The approach to criticality and critical exponents depend on the choice of the groundstate labelled by the Kac label $r = 1, 2, 3, \ldots$ or the height $b = 2, 4, 6, \ldots$.

- This underscores the existence of an infinite number of distinct critical exponents, order parameters and scaling operators — the theory is not rational!

Summary and Outlook

- In two dimensions, simple statistical systems with local degrees of freedom, such as the Ising model of a magnet, are *rational theories* with a *finite* number of *scaling operators*. The simplest such theories are the *minimal models* $\mathcal{M}(m, m')$ associated with the RSOS lattice models. The operator content and associated critical exponents are encoded in a *finite Kac table* of conformal dimensions.

- Two dimensional systems with nonlocal degrees of freedom, such as *polymers and percolation*, are not rational theories — they are *logarithmic theories* with an infinite number of scaling operators. The simplest such theories are the *logarithmic minimal models* $\mathcal{LM}(p, p')$. An infinite number of scaling operators and associated critical exponents are encoded in an *infinitely extended Kac table*.

- The logarithmic minimal models can be obtained as a limit of the minimal models

$$\lim_{m,m'\to\infty,\ \frac{m}{m'}\to\frac{p}{p'}} \mathcal{M}(m, m') = \mathcal{LM}(p, p'), \qquad 1 \le p < p', \quad p, p' \text{ coprime}$$

The logarithmic limit of certain GOPs $\mathcal{O}_{r,s}$ yield critical exponents $\beta_{r,s}$ associated with conformal weights $\Delta_{r,s}^{p,p'}$ of the logarithmic minimal models $\mathcal{LM}(p, p')$ in the infinitely extended Kac table.

- We conclude that *generalized models of polymers and percolation* are exactly solvable both *at criticality and off-criticality!*

- We described the integrable $\varphi_{1,3}$ off-critical perturbation but the $\varphi_{2,1}$ and $\varphi_{1,2}$ perturbations are also integrable by studying *dilute lattice models* (Warnaar et al 1992/94).

Logarithmic Ising and Yang-Lee Kac Tables

s							
s	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	$\frac{225}{16}$	$\frac{161}{16}$	$\frac{323}{48}$	$\frac{65}{16}$	$\frac{33}{16}$	$\frac{35}{48}$	$\cdots$
9	11	$\frac{15}{2}$	$\frac{14}{3}$	$\frac{5}{2}$	1	$\frac{1}{6}$	$\cdots$
8	$\frac{133}{16}$	$\frac{85}{16}$	$\frac{143}{48}$	$\frac{21}{16}$	$\frac{5}{16}$	$-\frac{1}{48}$	$\cdots$
7	6	$\frac{7}{2}$	$\frac{5}{3}$	$\frac{1}{2}$	0	$\frac{1}{6}$	$\cdots$
6	$\frac{65}{16}$	$\frac{33}{16}$	$\frac{35}{48}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{35}{48}$	$\cdots$
5	$\frac{5}{2}$	1	$\frac{1}{6}$	0	$\frac{1}{2}$	$\frac{5}{3}$	$\cdots$
4	$\frac{21}{16}$	$\frac{5}{16}$	$-\frac{1}{48}$	$\frac{5}{16}$	$\frac{21}{16}$	$\frac{143}{48}$	$\cdots$
3	$\frac{1}{2}$	0	$\frac{1}{6}$	1	$\frac{5}{2}$	$\frac{14}{3}$	$\cdots$
2	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{35}{48}$	$\frac{33}{16}$	$\frac{65}{16}$	$\frac{323}{48}$	$\cdots$
1	0	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{7}{2}$	6	$\frac{55}{6}$	$\cdots$
	1	2	3	4	5	6	r

s	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
10	$\frac{27}{5}$	$\frac{91}{40}$	$\frac{2}{5}$	$-\frac{9}{40}$	$\frac{2}{5}$	$\frac{91}{40}$	$\cdots$
9	4	$\frac{11}{8}$	0	$-\frac{1}{8}$	1	$\frac{27}{8}$	$\cdots$
8	$\frac{14}{5}$	$\frac{27}{40}$	$-\frac{1}{5}$	$\frac{7}{40}$	$\frac{9}{5}$	$\frac{187}{40}$	$\cdots$
7	$\frac{9}{5}$	$\frac{7}{40}$	$-\frac{1}{5}$	$\frac{27}{40}$	$\frac{14}{5}$	$\frac{247}{40}$	$\cdots$
6	1	$-\frac{1}{8}$	0	$\frac{11}{8}$	4	$\frac{63}{8}$	$\cdots$
5	$\frac{2}{5}$	$-\frac{9}{40}$	$\frac{2}{5}$	$\frac{91}{40}$	$\frac{27}{5}$	$\frac{391}{40}$	$\cdots$
4	0	$-\frac{1}{8}$	1	$\frac{27}{8}$	7	$\frac{95}{8}$	$\cdots$
3	$-\frac{1}{5}$	$\frac{7}{40}$	$\frac{9}{5}$	$\frac{187}{40}$	$\frac{44}{5}$	$\frac{567}{40}$	$\cdots$
2	$-\frac{1}{5}$	$\frac{27}{40}$	$\frac{14}{5}$	$\frac{247}{40}$	$\frac{54}{5}$	$\frac{667}{40}$	$\cdots$
1	0	$\frac{11}{8}$	4	$\frac{63}{8}$	13	$\frac{155}{8}$	$\cdots$
	1	2	3	4	5	6	r

Dense Polymer Virasoro Fusion Algebra

- The fundamental Virasoro fusion algebra of critical dense polymers $\mathcal{LM}(1,2)$ is

$$\Big\langle (2,1),(1,2)\Big\rangle \;=\; \Big\langle (r,1),(1,2k),\mathcal{R}_k;\;\; r,k\in\mathbb{N}\Big\rangle$$

- With the identifications $(k,2k')\equiv(k',2k)$, the fusion rules obtained empirically from the lattice are commutative, associative and agree with Gaberdiel and Kausch (1996)

$$(r,1)\otimes(r',1) \;=\; \bigoplus_{j=|r-r'|+1,\text{ by }2}^{r+r'-1} (j,1)$$

$$(1,2k)\otimes(1,2k') \;=\; \bigoplus_{j=|k-k'|+1,\text{ by }2}^{k+k'-1} \mathcal{R}_j$$

$$(1,2k)\otimes\mathcal{R}_{k'} \;=\; \bigoplus_{j=|k-k'|}^{k+k'} \delta^{(2)}_{j,\{k,k'\}}(1,2j)$$

$$\mathcal{R}_k\otimes\mathcal{R}_{k'} \;=\; \bigoplus_{j=|k-k'|}^{k+k'} \delta^{(2)}_{j,\{k,k'\}}\,\mathcal{R}_j$$

$$(r,1)\otimes(1,2k) \;=\; \bigoplus_{j=|r-k|+1,\text{ by }2}^{r+k-1} (1,2j) = (r,2k)$$

$$(r,1)\otimes\mathcal{R}_k \;=\; \bigoplus_{j=|r-k|+1,\text{ by }2}^{r+k-1} \mathcal{R}_j$$

s \ r	1	2	3	4	5	6	r
s	⋮	⋮	⋮	⋮	⋮	⋮	⋰
10	$\frac{63}{8}$	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	⋯
9	6	3	1	0	0	1	⋯
8	$\frac{35}{8}$	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	⋯
7	3	1	0	0	1	3	⋯
6	$\frac{15}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	⋯
5	1	0	0	1	3	6	⋯
4	$\frac{3}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	⋯
3	0	0	1	3	6	10	⋯
2	$-\frac{1}{8}$	$\frac{3}{8}$	$\frac{15}{8}$	$\frac{35}{8}$	$\frac{63}{8}$	$\frac{99}{8}$	⋯
1	0	1	3	6	10	15	⋯

$\mathcal{R}_k$ = indecomposable = $(1,2k-1)\oplus_i(1,2k+1)$,

$$\delta^{(2)}_{j,\{k,k'\}} = 2-\delta_{j,|k-k'|}-\delta_{j,k+k'}$$

$\mathcal{W}$-Extended Boundary Conditions

- Critical dense polymers in the $\mathcal{W}$-extended picture is identified with *symplectic fermions*.
- The integrable boundary condition associated to the $\mathcal{W}$-vacuum is

$$(1,1)_{\mathcal{W}} := \lim_{n\to\infty}(2n-1,1)\otimes(2n-1,1)\otimes(2n-1,1) = \bigoplus_{n=1}^{\infty}(2n-1)\,(2n-1,1)$$

- Using stability, the extended fusion $\hat{\otimes}$ is defined by

$$(1,1)_{\mathcal{W}}\,\hat{\otimes}\,(1,1)_{\mathcal{W}} := \lim_{n\to\infty}\Big(\frac{1}{(2n-1)^3}(2n-1,1)\otimes(2n-1,1)\otimes(2n-1,1)\otimes(1,1)_{\mathcal{W}}\Big) = (1,1)_{\mathcal{W}}$$

$$(2m-1,1)\otimes(1,1)_{\mathcal{W}} = (2m-1)\Big(\bigoplus_{n=1}^{\infty}(2n-1)\,(2n-1,1)\Big) = (2m-1)\,(1,1)_{\mathcal{W}}$$

- The $\mathcal{W}$-representation content is 4 $\mathcal{W}$-irreducible and 2 $\mathcal{W}$-reducible yet $\mathcal{W}$-indecomposable representations:

$$\begin{aligned}
(1,s)_{\mathcal{W}} &:= (1,s)\otimes(1,1)_{\mathcal{W}} = \bigoplus_{n=1}^{\infty}(2n-1)\,(2n-1,s), \qquad s=1,2\\
(2,s)_{\mathcal{W}} &:= \tfrac{1}{2}(2,s)\otimes(1,1)_{\mathcal{W}} = \bigoplus_{n=1}^{\infty}2n\,(2n,s), \qquad s=1,2\\
\hat{\mathcal{R}}_1 \equiv (\mathcal{R}_1)_{\mathcal{W}} &:= \mathcal{R}_1\otimes(1,1)_{\mathcal{W}} = \bigoplus_{n=1}^{\infty}(2n-1)\,\mathcal{R}_{2n-1}\\
\hat{\mathcal{R}}_0 \equiv (\mathcal{R}_2)_{\mathcal{W}} &:= \tfrac{1}{2}\mathcal{R}_2\otimes(1,1)_{\mathcal{W}} = \bigoplus_{n=1}^{\infty}2n\,\mathcal{R}_{2n}
\end{aligned}$$

- These give solutions to the BYBE since these equations are closed under fusions.

Symplectic Fermion $\mathcal{WLM}(1,2)$ Fusion Rules

- The $\mathcal{W}$-extended fusion rules follow from the Virasoro fusion rules. The extended fusion rules and characters agree with Gaberdiel and Runkel (2008):

$\hat\otimes$	0	1	$-\frac{1}{8}$	$\frac{3}{8}$	$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_1$
0	0	1	$-\frac{1}{8}$	$\frac{3}{8}$	$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_1$
1	1	0	$\frac{3}{8}$	$-\frac{1}{8}$	$\hat{\mathcal{R}}_1$	$\hat{\mathcal{R}}_0$
$-\frac{1}{8}$	$-\frac{1}{8}$	$\frac{3}{8}$	$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_1$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2(-\frac{1}{8})+2(\frac{3}{8})$
$\frac{3}{8}$	$\frac{3}{8}$	$-\frac{1}{8}$	$\hat{\mathcal{R}}_1$	$\hat{\mathcal{R}}_0$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2(-\frac{1}{8})+2(\frac{3}{8})$
$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_0$	$\hat{\mathcal{R}}_1$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2\hat{\mathcal{R}}_0+2\hat{\mathcal{R}}_1$	$2\hat{\mathcal{R}}_0+2\hat{\mathcal{R}}_1$
$\hat{\mathcal{R}}_1$	$\hat{\mathcal{R}}_1$	$\hat{\mathcal{R}}_0$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2(-\frac{1}{8})+2(\frac{3}{8})$	$2\hat{\mathcal{R}}_0+2\hat{\mathcal{R}}_1$	$2\hat{\mathcal{R}}_0+2\hat{\mathcal{R}}_1$

Example: Consider the extended fusion rule $1\,\hat\otimes\,1=0$:

$$\begin{aligned}(2,1)_{\mathcal{W}}\,\hat\otimes\,(2,1)_{\mathcal{W}} &:= \Big(\tfrac{1}{2}(2,1)\otimes(1,1)_{\mathcal{W}}\Big)\hat\otimes\Big(\tfrac{1}{2}(2,1)\otimes(1,1)_{\mathcal{W}}\Big)\\ &= \tfrac{1}{4}\Big((2,1)\otimes(2,1)\Big)\otimes\Big((1,1)_{\mathcal{W}}\,\hat\otimes\,(1,1)_{\mathcal{W}}\Big)\\ &= \tfrac{1}{4}\Big((1,1)\oplus(3,1)\Big)\otimes(1,1)_{\mathcal{W}} = \tfrac{1}{4}(1+3)(1,1)_{\mathcal{W}} = (1,1)_{\mathcal{W}}\end{aligned}$$

This follows since the extended vacuum has the stability property

$$(2m-1,1)\otimes(1,1)_{\mathcal{W}} = (2m-1)\Big(\bigoplus_{n=1}^{\infty}(2n-1)\,(2n-1,1)\Big) = (2m-1)\,(1,1)_{\mathcal{W}}$$

Representation Content of $\mathcal{WLM}(p,p')$

	$\mathcal{WLM}(p,p')$	Symplectic Fermions	$\mathcal{WLM}(1,p)$	Critical Percolation
$\mathcal{W}$-reps	$6pp'-2p-2p'$	6	$4p-2$	26
Rank 1	$2p+2p'-2$	4	$2p$	8
Rank 2	$4pp'-2p-2p'$	2	$2p-2$	14
Rank 3	$2(p-1)(p'-1)$	0	0	4
$\mathcal{W}$-irred chars	$2pp'+\frac{1}{2}(p-1)(p'-1)$	4	$2p$	13

- Kac tables of 4 and 13 $\mathcal{W}$-irreducible characters for symplectic fermions and critical percolation:

s \ r	1	2
2	$-\frac{1}{8}$	$\frac{3}{8}$
1	0	1

s \ r	1	2
3	$\frac{1}{3},\frac{10}{3}$	$-\frac{1}{24},\frac{35}{24}$
2	$1,5$	$\frac{1}{8},\frac{21}{8}$
1	$(0)\,2,7$	$\frac{5}{8},\frac{33}{8}$

- The irreducible representation (0) with character $\hat{\chi}_0(q)=1$ has no conjugate boundary condition. Similarly for $(1),(2),(5),(7)$.
- The fusion algebra of critical percolation has no identity!
- The usual $\frac{1}{2}(p-1)(p'-1)$ rational minimal representations re-emerge!

$\mathcal{W}$-Ireducible Characters of Critical Percolation

- $\mathcal{W}$-irreducible representations:

$$\hat{\chi}_{\frac{1}{3}}(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} (2k-1) q^{3(4k-3)^2/8} \qquad \hat{\chi}_{\frac{21}{8}}(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} 2k\, q^{(6k-2)^2/6}$$

$$\hat{\chi}_{\frac{10}{3}}(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} 2k\, q^{3(4k-1)^2/8} \qquad \hat{\chi}_{\frac{33}{8}}(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} 2k\, q^{(6k-1)^2/6}$$

$$\hat{\chi}_{\frac{1}{8}}(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} (2k-1)\, q^{(6k-5)^2/6} \qquad \hat{\chi}_{-\frac{1}{24}}(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} (2k-1)\, q^{(6k-6)^2/6}$$

$$\hat{\chi}_{\frac{5}{8}}(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} (2k-1)\, q^{(6k-4)^2/6} \qquad \hat{\chi}_{\frac{35}{24}}(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} 2k\, q^{(6k-3)^2/6}$$

- From subfactors of $\mathcal{W}$-reducible yet $\mathcal{W}$-indecomposable representations:

$$\hat{\chi}_0(q) = 1$$

$$\hat{\chi}_1(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} k^2 \Big[q^{(12k-7)^2/24} - q^{(12k+1)^2/24} \Big]$$

$$\hat{\chi}_2(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} k^2 \Big[q^{(12k-5)^2/24} - q^{(12k-1)^2/24} \Big] \qquad \eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n)$$

$$\hat{\chi}_5(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} k(k+1) \Big[q^{(12k-1)^2/24} - q^{(12k+7)^2/24} \Big]$$

$$\hat{\chi}_7(q) = \frac{1}{\eta(q)} \sum_{k\in\mathbb{Z}} k(k+1) \Big[q^{(12k+1)^2/24} - q^{(12k+5)^2/24} \Big]$$

- These agree with Feigin, Gainutdinov, Semikhatov and Tipunin (2005).

Summary of Virasoro and $\mathcal{W}$-Extended Pictures

- Representation Content:

Reps	Dense Polymers/ Symp Fermions	$\mathcal{LM}(1,p)$	Percolation	$\mathcal{LM}(p,p')$
Vir	∞	∞	∞	∞
$\mathcal{W}$	6	$4p-2$	26	$6pp'-2p-2p'$

- Empirical Virasoro fusion rules for $\mathcal{LM}(p,p')$:

Checks:
1. $\mathcal{LM}(p,p')$ fusion rules agree with level-by-level fusion rules of Eberle and Flohr (2006) using the Nahm (1994) algorithm.
2. Vertical sub-fusion algebras agree with Read and Saleur (2007).
3. Associativity.

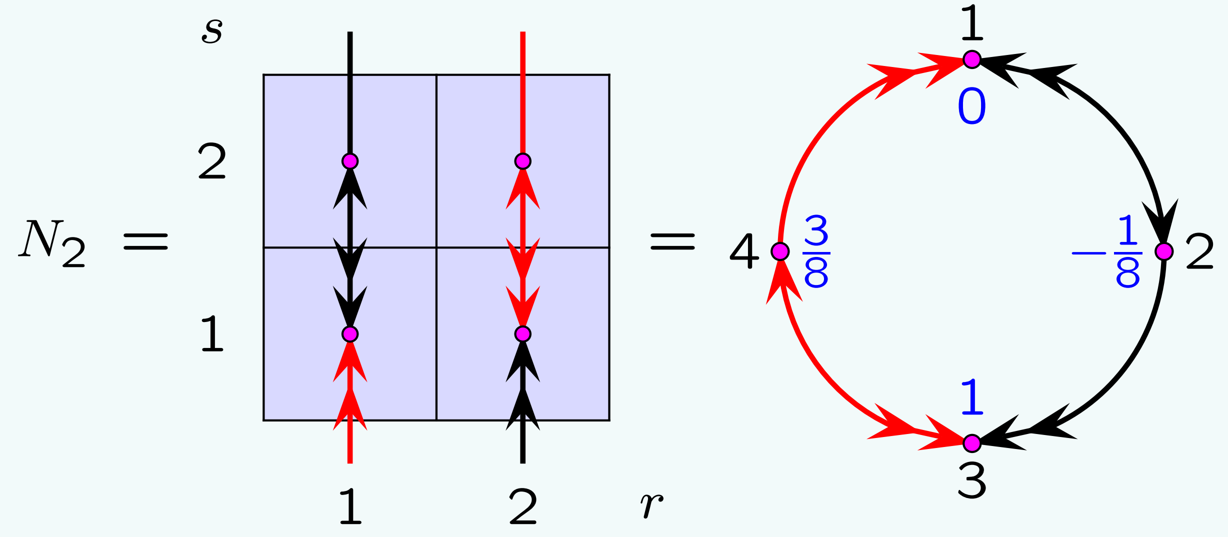
- Inferred $\mathcal{W}$-algebra fusion rules for $\mathcal{WLM}(p,p')$:

Checks:
1. $\mathcal{WLM}(1,p)$ fusion rules agree with Gaberdiel and Kausch (1996) and Gaberdiel and Runkel (2008).
2. $\mathcal{WLM}(p,p')$ characters agree with Feigin et al (2006).
3. Associativity.
4. $\mathcal{WLM}(2,3)$ percolation fusion confirmed "from within CFT" by Gaberdiel, Runkel and Wood (2009).

Symplectic Fermion $\mathcal{WLM}(1,2)$ Grothendieck Ring

- The partition functions are blind to indecomposability.
- Symplectic fermions has 4 rank-1 irreducible representations with distinct characters: $\hat{\chi}_0(q)$, $\hat{\chi}_{-1/8}(q)$, $\hat{\chi}_1(q)$, $\hat{\chi}_{3/8}(q)$ with $\chi[\hat{\mathcal{R}}_0](q) = \chi[\hat{\mathcal{R}}_1](q) = 2\hat{\chi}_0(q) + 2\hat{\chi}_1(q)$.
- The Grothendieck ring is the quotient fusion algebra under identifications modulo indecomposable structures. Informally, it is the *"fusion ring of irreducible characters"*.

$\hat{\otimes}$	0	$-\frac{1}{8}$	1	$\frac{3}{8}$
0	0	$-\frac{1}{8}$	1	$\frac{3}{8}$
$-\frac{1}{8}$	$-\frac{1}{8}$	$2(0)+2(1)$	$\frac{3}{8}$	$2(0)+2(1)$
1	1	$\frac{3}{8}$	0	$-\frac{1}{8}$
$\frac{3}{8}$	$\frac{3}{8}$	$2(0)+2(1)$	$-\frac{1}{8}$	$2(0)+2(1)$



- The four fusion matrices of the graph fusion algebra can be read off:

$$N_1 = N_{(0)} = I,\ \ N_2 = N_{(-\frac{1}{8})} = \begin{pmatrix} 0&1&0&0\\ 2&0&2&0\\ 0&0&0&1\\ 2&0&2&0 \end{pmatrix},\ \ N_3 = N_{(1)} = \begin{pmatrix} 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0 \end{pmatrix},\ \ N_4 = N_{(\frac{3}{8})} = \begin{pmatrix} 0&0&0&1\\ 2&0&2&0\\ 0&1&0&0\\ 2&0&2&0 \end{pmatrix}$$

$$N_i N_j = \sum_k N_{ij}{}^k N_k, \qquad \text{1-dim rep: } N_{r,s} = \cos^{r-1} p\theta \, \frac{\sin s\theta}{\sin\theta} = q\text{-dims}, \qquad \theta = \pi/p$$

$$N_2 N_1 = N_2, \quad N_2 N_2 = 2N_1 + 2N_3, \quad \text{etc} \qquad N_{r,s} = f_{r,s}(N_2), \qquad f_{r,s}(x) = T_p^{r-1}(\tfrac{x}{2}) U_{s-1}(\tfrac{x}{2})$$

- The fusion matrices are mutually commuting but not symmetric and *not diagonalizable!*

Modular Matrix of $\mathcal{WLM}(1,p)$

- For rational theories, the modular matrix "*diagonalizes the fusion rules*".
- But here the $2p$ irreducible characters do not close under modular transformations

$$\hat{\chi}_{r,s}(q) \;=\; \frac{1}{\eta(q)}\sum_{j\in\mathbb{Z}}(2j+r)q^{p(j+\frac{rp-s}{2p})^2}, \qquad r=1,2; \quad s=1,2,\ldots,p$$

- Closure is achieved by adding $p-1$ *pseudocharacters* [Feigin Et Al (2006)]

$$\hat{\chi}_{0,b}(q) \;=\; i\tau[b\,\hat{\chi}_{1,p-b}(q)-(p-b)\hat{\chi}_{2,b}(q)], \quad b=1,2,\ldots,p-1, \qquad q \;=\; e^{2\pi i\tau}$$

- The $3p-1$ dimensional S-matrix is $\quad (S^2=I,\; S\neq S^T)$

$$S=\begin{pmatrix} S_{r,s}^{r',s'} & S_{r,s}^{0,b'} \\ S_{0,b}^{r',s'} & S_{0,b}^{0,b'}\end{pmatrix} = \begin{pmatrix} \frac{(2-\delta_{s',p})(-1)^{rs'+r's+rr'p}s\cos\frac{ss'\pi}{p}}{p\sqrt{2p}} & \frac{2(-1)^{rb'}\sin\frac{sb'\pi}{p}}{p\sqrt{2p}} \\ \frac{2(-1)^{r'b}(p-s')\sin\frac{bs'\pi}{p}}{\sqrt{2p}} & 0\end{pmatrix}, \qquad S_{r,s}^{1,p-b} \;=\; S_{r,s}^{2,b}$$

- Gaberdiel and Runkel introduce the *improper* τ-dependent $2p$ dimensional "S-matrix"

$$\mathcal{S}(\tau) \;=\; \big(S_{r,s}^{r',s'}\big)-i\tau\big(T_{r,s}^{r',s'}\big), \qquad T_{r,s}^{r',s'}=\frac{2(-1)^{rs'+r+rr'p}(p-s')\sin\frac{ss'\pi}{p}}{p\sqrt{2p}}$$

- The matrices S and $\mathcal{S}(\tau)$ contain the same "modular data" since

$$(p-b)T_{r,s}^{1,p-b} \;=\; -bT_{r,s}^{2,b}$$

$$T_{r,s}^{1,b'} \;=\; -(p-b')S_{r,s}^{0,p-b'}, \qquad T_{r,s}^{2,b'} \;=\; (p-b')S_{r,s}^{0,b'}, \qquad T_{r,s}^{r',p} \;=\; 0$$

Jordan Form of the Fundamental Matrix

- The $p+1$ distinct eigenvalues of the fundamental N_2 are

$$\beta_j \;=\; 2\cos\theta_j, \qquad \theta_j \;=\; \frac{j\pi}{p}, \qquad j=0,1,\ldots,p$$

- The Jordan canonical form of N_2 consists of:

	Block	Eigenvalue (q-dim)	Eigenvector
one rank-1 cell:	(2),	$\beta_0 = \frac{S^{1,p}_{1,2}}{S^{1,p}_{1,1}} = 2$	$\boldsymbol{v}_0$
$p-1$ rank-2 cells:	$\begin{pmatrix}\beta_b & 1\\ 0 & \beta_b\end{pmatrix}$,	$\beta_b = \frac{S^{0,b}_{1,2}}{S^{0,b}_{1,1}}, \quad b=1,2,\ldots,p-1$	$\boldsymbol{v}_b$
one rank-1 cell:	(-2),	$\beta_p = \frac{S^{2,p}_{1,2}}{S^{2,p}_{1,1}} = -2$	$\boldsymbol{v}_p$

- Eigenvectors $\boldsymbol{v}_j$ and generalized eigenvectors $\boldsymbol{w}_j$ are given by

$$[\boldsymbol{v}_0]_{r,s} \;=\; S^{1,p}_{r,s}, \qquad [\boldsymbol{v}_p]_{r,s} \;=\; S^{2,p}_{r,s}, \qquad [\boldsymbol{v}_b]_{r,s} \;=\; S^{0,b}_{r,s}, \qquad [\boldsymbol{w}_b]_{r,s} \;=\; S^{2,b}_{r,s}$$

with the Jordan chain

$$N_2\big(\boldsymbol{v}_b \big| \boldsymbol{w}_b\big) \;=\; \big(\boldsymbol{v}_b \big| \boldsymbol{w}_b\big)\begin{pmatrix}\beta_b & -2\sin\theta_b\\ 0 & \beta_b\end{pmatrix} \;=\; \big(\boldsymbol{v}_b \big| \boldsymbol{w}_b\big)\,2\cos\begin{pmatrix}\theta_b & 1\\ 0 & \theta_b\end{pmatrix}, \qquad b=1,2,\ldots,p-1$$

since

$$f\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix} \;=\; \begin{pmatrix}f(\lambda) & f'(\lambda)\\ 0 & f(\lambda)\end{pmatrix}$$

Verlinde Formula for $\mathcal{WLM}(1,p)$

- The modular data simultaneously brings the fusion matrices $N_j = N_{r,s}$ to Jordan form with

$$Q^{-1} N_{r,s}\, Q \;=\; f_{r,s}\Big(\mathrm{diag}\big(2, \underbrace{\ldots, 2\cos\begin{pmatrix}\theta_b & 1\\ 0 & \theta_b\end{pmatrix}, \ldots,}_{p-1} -2\big)\Big)$$

This is not strict Jordan canonical form. The matrices Q and Q^{-1} are

$$Q = \left(S^{1,p}_{r,s}\,\Big|\ldots\Big|\, S^{0,b}_{r,s}\quad S^{2,b}_{r,s}\,\Big|\ldots\Big|\, S^{2,p}_{r,s}\right), \qquad Q^{-1} = \left(\frac{S^{1,p}_{1,p}}{S^{1,p}_{1,1}} S^{r,s}_{1,p}\,\Bigg|\cdots\Bigg|\, S^{r,s}_{0,b}\quad \frac{S^{1,p}_{1,p}}{S^{1,p}_{1,b}} S^{r,s}_{2,b}\,\Bigg|\cdots\Bigg|\, \frac{S^{1,p}_{1,p}}{S^{1,p}_{1,1}} S^{r,s}_{2,p}\right)^T$$

- The Verlinde formula is given by the "*spectral decomposition*" of the fusion matrices

$$\begin{aligned}
[N_{r,s}]^{r'',s''}_{r',s'} \;=\; & \Big[S^{1,p}_{r',s'} F^{1,p}_{r,s} S^{r'',s''}_{1,p} + \sum_{b=1}^{p-1} S^{2,b}_{r',s'} F^{2,b}_{r,s} S^{r'',s''}_{2,b} + S^{2,p}_{r',s'} F^{2,p}_{r,s} S^{r'',s''}_{2,p}\Big] \\
& + \Big[\sum_{b=1}^{p-1} S^{0,b}_{r',s'} F^{0,b}_{r,s} S^{r'',s''}_{0,b}\Big] + \Big[\sum_{b=1}^{p-1} S^{0,b}_{r',s'} F^{0,b;2,b}_{r,s} S^{r'',s''}_{2,b}\Big]
\end{aligned}$$

$$F^{1,p}_{r,s} = \frac{S^{1,p}_{1,p} S^{1,p}_{r,s}}{(S^{1,p}_{1,1})^2}, \qquad F^{2,b}_{r,s} = \frac{S^{1,p}_{1,p} S^{0,b}_{r,s}}{S^{1,p}_{1,b} S^{0,b}_{1,1}}, \qquad F^{2,p}_{r,s} = \frac{S^{1,p}_{1,p} S^{2,p}_{r,s}}{S^{1,p}_{1,1} S^{2,p}_{1,1}}$$

$$F^{0,b}_{r,s} = \frac{S^{0,b}_{r,s}}{S^{0,b}_{1,1}}, \qquad F^{0,b;2,b}_{r,s} = \frac{S^{1,p}_{1,p}(S^{0,b}_{1,1} S^{2,b}_{r,s} - S^{2,b}_{1,1} S^{0,b}_{r,s})}{S^{1,p}_{1,b}(S^{0,b}_{1,1})^2}$$

- After some manipulation, this agrees with a similar formula in terms of the τ-dependent S-matrix conjectured by Gaberdiel and Runkel (2007).

$\mathcal{W}$-Projective Representations

- A $\mathcal{W}$-projective representation is a "*maximal*" $\mathcal{W}$-indecomposable representation in the sense that it does not appear as a subfactor of any other $\mathcal{W}$-indecomposable representation.
- Symplectic fermions has 4 projective representations $-1/8$, $3/8$, $\hat{\mathcal{R}}_0$ and $\hat{\mathcal{R}}_1$ with 3 distinct characters $\hat{\chi}_{-1/8}(q)$, $\hat{\chi}_{3/8}(q)$ and $\chi[\hat{\mathcal{R}}_0](q) = \chi[\hat{\mathcal{R}}_1](q) = 2\hat{\chi}_0(q) + 2\hat{\chi}_1(q)$.
- The $\mathcal{W}$-projective representations form a closed sub-fusion algebra *Proj* (p,p') of the $\mathcal{WLM}(p,p')$ fusion algebra.
- The $\mathcal{W}$-projective representation content is:

	Reps	$\mathcal{WLM}(p,p')$	Symplectic Fermions	Critical Percolation
$\mathcal{W}$-proj reps	$\hat{\mathcal{R}}_{\kappa p,p'}^{r,s}$	$2pp'$	4	12
Rank 1	$\hat{\mathcal{R}}_{\kappa p,p'}^{0,0} \equiv (\kappa p, p')_{\mathcal{W}}$	2	2	2
Rank 2	$\hat{\mathcal{R}}_{\kappa p,p'}^{a,0}$, $\hat{\mathcal{R}}_{p,\kappa p'}^{0,b}$	$2(p+p'-2)$	2	6
Rank 3	$\hat{\mathcal{R}}_{\kappa p,p'}^{a,b}$	$2(p-1)(p'-1)$	0	4
$\mathcal{W}$-proj chars	$\varkappa_k$	$\frac{1}{2}(p+1)(p'+1)$	3	6

$$(\kappa p, p')_{\mathcal{W}} = (p, \kappa p')_{\mathcal{W}}, \qquad \hat{\mathcal{R}}_{\kappa p,p'}^{a,b} = \hat{\mathcal{R}}_{p,\kappa p'}^{a,b}$$

$$\kappa = 1,2; \quad a = 1,2,\ldots,p-1; \quad b = 1,2,\ldots,p'-1; \quad k = 1,2,\ldots,\tfrac{1}{2}(p+1)(p'+1)$$
$$r = 0,1,\ldots,p; \qquad s = 0,1,\ldots,p'$$

$\mathcal{W}$-Projective Characters and Grothendieck Ring

- The characters of the $2pp'$ $\mathcal{W}$-projective representations agree with Feigin et al (2006)

$$\begin{aligned}
\varkappa^{0,0}_{\kappa p,p'}(q) &\equiv \varkappa\big[\hat{\mathcal{R}}^{0,0}_{\kappa p,p'}\big](q) &=& \frac{1}{\eta(q)}\sum_{k\in\mathbb{Z}}(2k-2+\kappa)q^{((2k-2+\kappa)-1)^2pp'/4}\\
\varkappa^{a,0}_{\kappa p,p'}(q) &\equiv \varkappa\big[\hat{\mathcal{R}}^{a,0}_{\kappa p,p'}\big](q) &=& \frac{2}{\eta(q)}\sum_{k\in\mathbb{Z}}q^{(a+(2k-1+\kappa)p)^2p'/4p}\\
\varkappa^{0,b}_{p,\kappa p'}(q) &\equiv \varkappa\big[\hat{\mathcal{R}}^{0,b}_{p,\kappa p'}\big](q) &=& \frac{2}{\eta(q)}\sum_{k\in\mathbb{Z}}q^{(b+(2k-1+\kappa)p')^2p/4p'}\\
\varkappa^{a,b}_{\kappa p,p'}(q) &\equiv \varkappa[\hat{\mathcal{R}}^{a,b}_{\kappa p,p'}](q) &=& \frac{2}{\eta(q)}\sum_{k\in\mathbb{Z}}\Big(q^{(ap'-bp+(2k+1-\kappa)pp')^2/4pp'}+q^{(ap'+bp+(2k+1-\kappa)pp')^2/4pp'}\Big)
\end{aligned}$$

- Only $\frac{1}{2}(p+1)(p'+1)$ of these are linearly independent because of the character identities

$$\varkappa^{a,0}_{p,p'}(q) = \varkappa^{p-a,0}_{2p,p'}(q), \qquad \varkappa^{0,b}_{p,p'}(q) = \varkappa^{0,p'-b}_{p,2p'}(q), \qquad \varkappa^{a,b}_{(3-\kappa)p,p'}(q) = \varkappa^{p-a,b}_{\kappa p,p'}(q) = \varkappa^{a,p'-b}_{\kappa p,p'}(q)$$

- The $\mathcal{W}$-projective fusion algebra *Proj* (p,p') possesses a Grothendieck ring $\mathcal{PG}(p,p')$ corresponding to the $\frac{1}{2}(p+1)(p'+1)$ independent $\mathcal{W}$-projective characters:

$$\begin{aligned}
\mathcal{PG}(p,p') &= \Big\langle \varkappa_k(q)\Big|_{k=1}^{\frac{1}{2}(p+1)(p'+1)}\Big\rangle\\
&= \Big\langle \varkappa^{0,0}_{p,p'}(q),\ \varkappa^{0,0}_{2p,p'}(q),\ \varkappa^{a,0}_{p,p'}(q)\Big|_{a=1}^{p-1},\ \varkappa^{0,b}_{p,p'}(q)\Big|_{b=1}^{p'-1},\ \varkappa^{a,b}_{p,p'}(q)\Big|_{ap'+bp\le pp'}\Big\rangle
\end{aligned}$$

Verlinde Formula and Graph Fusion Algebra

- The modular S matrix of these characters [Feigin et al (2006)] satisfies $S^2 = I$, $S^T \neq S$.
- The modular matrix and quantum dimensions of $\mathcal{PG}(1,p)$ are

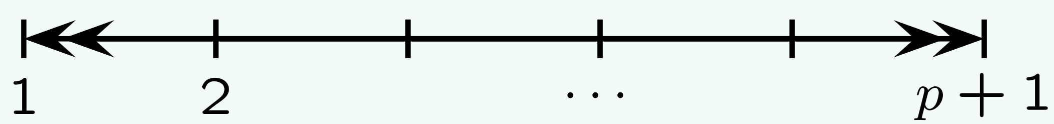
$$\sqrt{2p}\, S_{im} = \frac{S_{im}}{S_{1m}} = (2 - \delta_{i,1} - \delta_{i,p+1}) \cos\frac{(i-1)(m-1)\pi}{p}, \qquad i,m = 1,2,\ldots,p+1$$

- This modular matrix diagonalizes our fusion rules! Specifically, the conformal partition functions and Verlinde formula for the projective Grothendieck ring $\mathcal{PG}(p,p')$ are given by

$$Z_{i|j}(q) = \sum_{k=1}^{\frac{1}{2}(p+1)(p'+1)} N_{ij}{}^{k} (F\varkappa)_k(q), \qquad N_{ij}{}^{k} = \sum_{m=1}^{\frac{1}{2}(p+1)(p'+1)} \frac{S_{im}S_{jm}S_{mk}}{S_{1m}}, \qquad N_1 = I$$

- The fundamental fusion matrix of $\mathcal{PG}(1,p)$ is

$$N_2 = \begin{pmatrix} 0 & 1 & & & & & \\ 2 & 0 & 1 & & & & \\ & 1 & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & 1 & \\ & & & & 1 & 0 & 2 \\ & & & & & 1 & 0 \end{pmatrix} : \qquad A_{(1,p)} =$$



- For symplectic fermions $\mathcal{PG}(1,2)$, the graph fusion matrices (cf. Ising) are

$$N_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad N_2 = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} = F, \qquad N_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

- For $\mathcal{PG}(p,p')$, the fundamental fusion graph is given by a coset-type graph.

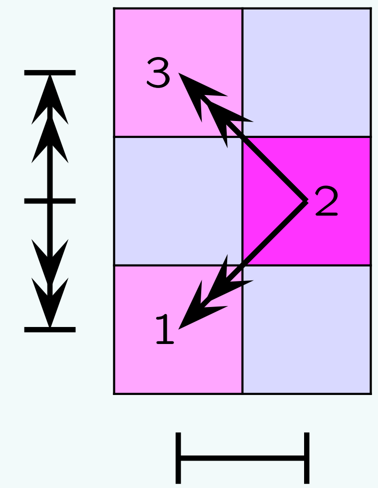
Projective Grothendieck Kac Tables

- The conformal weights of the projective Grothendieck characters of $\mathcal{PG}(p,p')$ are

$$\Delta_{r,s} = \frac{(p'r-ps)^2-(p-p')^2}{4pp'}, \qquad r=0,1,\ldots,p; \quad s=0,1,\ldots,p'$$

Symp Fermions:

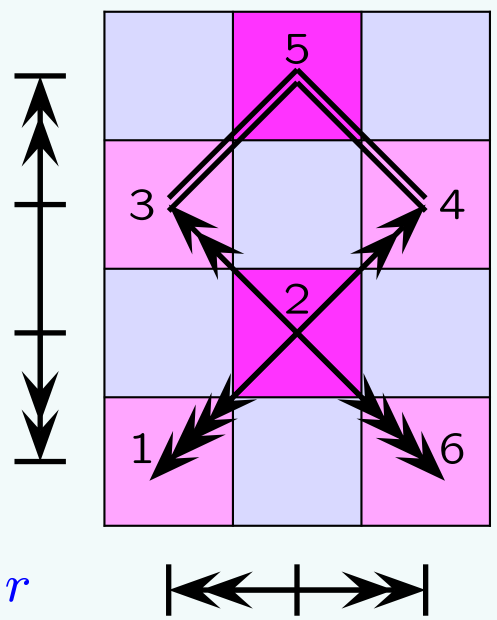
s \ r	0	1
2	$\frac{3}{8}$	$-\frac{1}{8}$
1	0	0
0	$-\frac{1}{8}$	$\frac{3}{8}$



$$N_2 = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} = F$$

Percolation:

s \ r	0	1	2
3	$\frac{35}{24}$	$\frac{1}{3}$	$-\frac{1}{24}$
2	$\frac{5}{8}$	0	$\frac{1}{8}$
1	$\frac{1}{8}$	0	$\frac{5}{8}$
0	$-\frac{1}{24}$	$\frac{1}{3}$	$\frac{35}{24}$

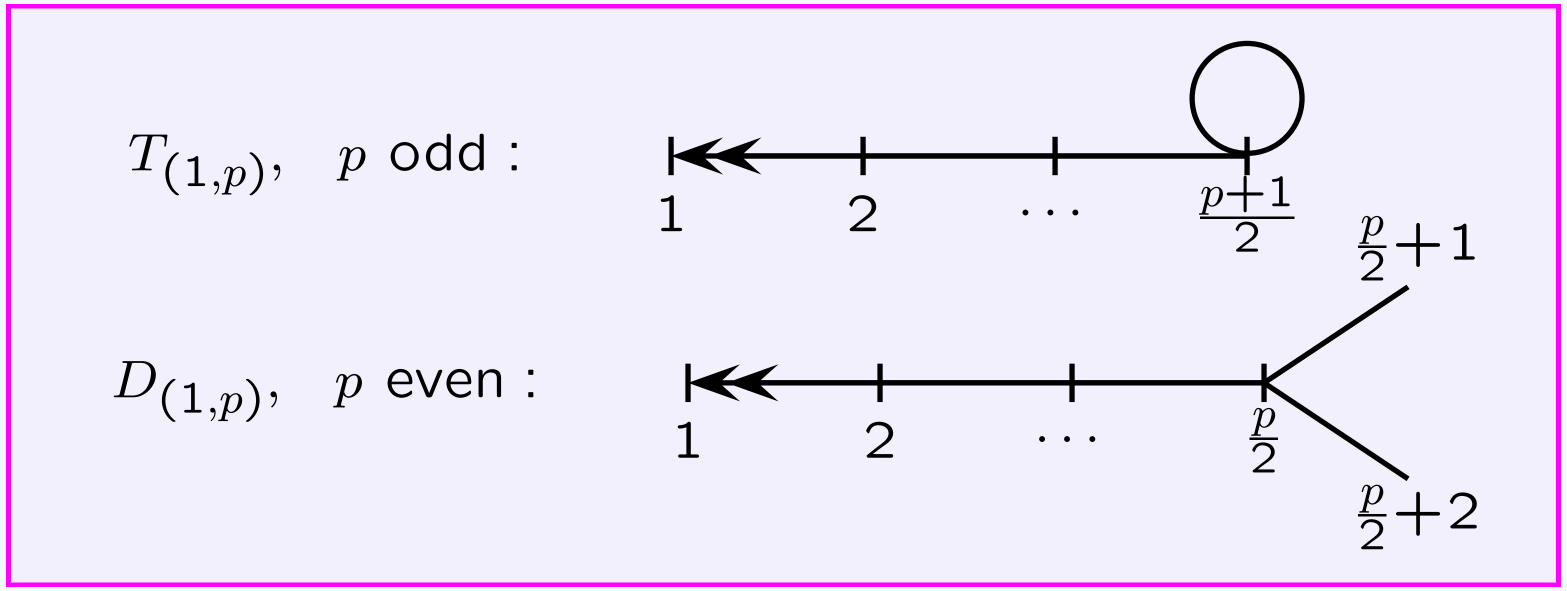


$$N_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & 0 & 2 & 2 & 0 & 4 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 4 & 0 & 4 & 4 & 0 & 4 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ 2 & 0 & 2 & 2 & 0 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= N_2 + N_5$$

A-D-E-T

- A $\mathbb{Z}_2$ folding or orbifold of the $A_{(1,p)}$ graphs gives T or D type graphs:



- Indeed, Feigin et al (2006) have found A, D and E_6 modular invariant sesquilinear forms in the characters $\varkappa_k(q) = \varkappa_{r,s}(q)$.

- This leads to some intriguing open questions:

1. Is there an A-D-E classification of these logarithmic Verlinde fusion graphs a la Behrend, Pearce, Petkova and Zuber?

2. Is there a corresponding A-D-E classification of the logarithmic modular invariant sesquilinear forms a la Cappelli, Itzykson and Zuber?

3. Is there a logarithmic coset construction a la Goddard, Kent and Olive?

4. Are there corresponding D and E logarithmic minimal models on the lattice?

Summary

- Representation Content:

Reps	Dense Polymers/ Symp Fermions	$\mathcal{LM}(1,p)$	Percolation
Vir	∞	∞	∞
$\mathcal{W}$	6	$4p-2$	26
$\mathcal{W}$ Grothendieck	4	$2p$	?
Proj	4	$2p$	12
Proj Grothendieck	3	$p+1$	6

- Grothendieck ring and Verlinde formulas for $\mathcal{WLM}(1,p)$:

1. The $\mathcal{W}$ Grothendieck ring is described by a simple (but non-diagonalizable) graph fusion algebra.
2. The modular data simultaneously brings the fusion matrices to Jordan form.
3. The resulting Verlinde formulas agree with Gaberdiel and Runkel (2007).

- Projective Grothendieck ring and Verlinde formulas for $\mathcal{WLM}(p,p')$:

1. Projective characters agree with Feigin et al (2006). The Grothendieck rings are described by simple graph fusion algebras.
2. Feigin et al modular S matrix diagonalizes our projective Grothendieck fusion rules!
3. Verlinde formulas and graph fusion algebras suggest an A-D-E classification.

Chiral Symplectic Fermions (Kausch 1995)

- The central charge of symplectic fermions is $c = -2$ and the stress-energy tensor is

$$T(z) = \sum_{n\in\mathbb{Z}} L_n \, z^{-n-2} = \tfrac{1}{2} d_{\alpha\beta} :\chi^{\alpha}(z)\chi^{\beta}(z):$$

where $d_{\alpha\beta}$ is the inverse of the anti-symmetric tensor $d^{\alpha\beta}$ with $\alpha, \beta = \pm$.

- The chiral algebra $\mathcal{W}$ is generated by a two-component fermion field

$$\chi^{\alpha}(z) = \sum_{n\in\mathbb{Z}} \chi^{\alpha}_n \, z^{-n-1}, \qquad \alpha = \pm$$

of conformal weight $\Delta = 1$. The modes satisfy the anticommutation relations

$$\{\chi^{\alpha}_m, \chi^{\beta}_n\} = m \, d^{\alpha\beta} \, \delta_{m,-n}$$

- Alternatively, the extended symmetry algebra $\mathcal{W}$ is generated by the Virasoro modes L_n and the modes of a triplet of weight 3 fields W^a_n.

Virasoro Representations and L_0

- In the continuum scaling limit, the transfer matrices give rise to a representation of the Virasoro algebra. Only L_0 is readily accessible from the lattice

$$\boldsymbol{D}(u) \sim e^{-u\mathcal{H}}, \qquad -\mathcal{H} \mapsto L_0 - \frac{c}{24}, \qquad Z_{r,s}(q) = \mathrm{Tr}\,\boldsymbol{D}(u)^P \mapsto q^{-c/24}\,\mathrm{Tr}\,q^{L_0} = \chi_{r,s}(q)$$

Type	Irreducible	Fully Reducible	Reducible yet Indecomposable	Decomposable
L_n	$\begin{pmatrix} \textcolor{red}{\blacksquare} \end{pmatrix}$	$\begin{pmatrix} \textcolor{red}{\blacksquare} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \textcolor{red}{\blacksquare} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \textcolor{red}{\blacksquare} \end{pmatrix}$	$\begin{pmatrix} \textcolor{red}{\blacksquare} & \textcolor{gray}{\blacksquare} \\ \mathbf{0} & \textcolor{red}{\blacksquare} \end{pmatrix}$	$\begin{pmatrix} \textcolor{red}{\blacksquare} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \textcolor{red}{\blacksquare} & \textcolor{gray}{\blacksquare} \\ \mathbf{0} & \mathbf{0} & \textcolor{red}{\blacksquare} \end{pmatrix}$
L_0	Diagonalizable	Diagonalizable	Jordan Cells of Rank ≥ 2	Jordan Cells

- **Rational Theories:**

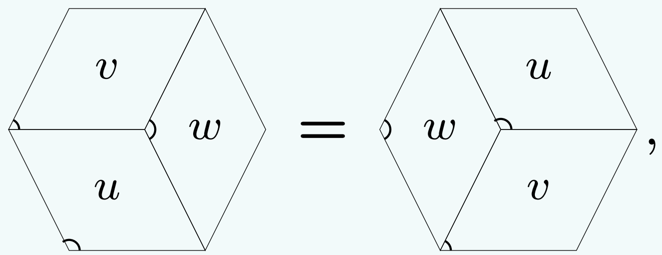
Irreducible representations are the building blocks for fusion. Fusion closes on the irreducible representations.

- **Logarithmic Theories:**

Kac representations are the building blocks for fusion. Higher rank indecomposable representations arise from fusing Kac representations.

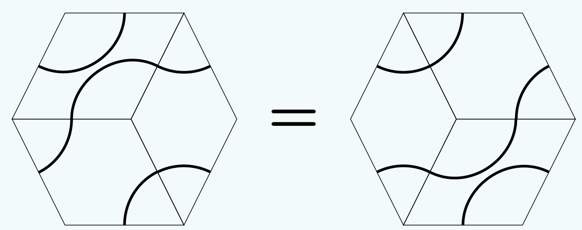
Integrability I: Yang-Baxter Equation (YBE)

- The YBE express the equality of two planar 3-tangles ($w = v - u$)

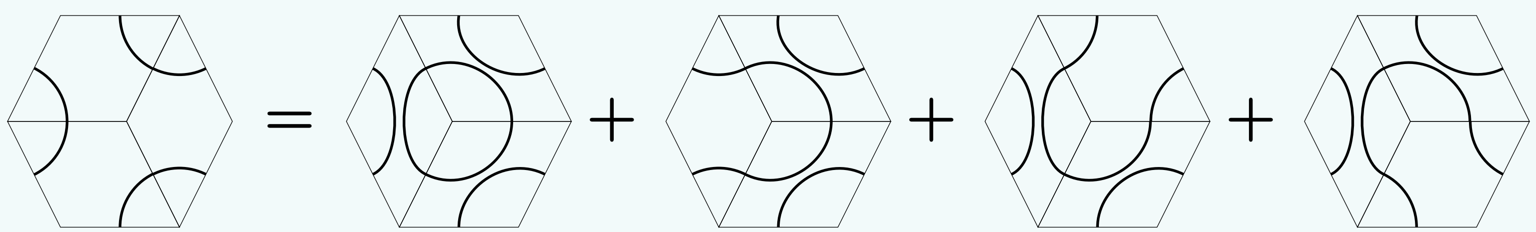


$$X_j(w)X_{j+1}(u)X_j(v) \;=\; X_{j+1}(v)X_j(u)X_{j+1}(w)$$

- The five possible connectivities of the external nodes give the diagrammatic equations



$\times\ 3\ \ (120^\circ$ rotations)



$\times\ 2\ \ (180^\circ$ rotations)

- The first equation is trivial. The second equation follows from the identity

$$\begin{aligned} s_1(-u)s_0(v)s_1(-w) \;&=\; \beta\, s_0(u)s_1(-v)s_0(w) \;+\; s_0(u)s_1(-v)s_1(-w) \\ &\quad + \; s_1(-u)s_1(-v)s_0(w) \;+\; s_0(u)s_0(v)s_0(w) \end{aligned}$$

$$s_r(u) = \frac{\sin(u + r\lambda)}{\sin\lambda}, \qquad \beta \;=\; 2\cos\lambda \;=\; \text{loop fugacity}$$