Renormalization and universality in non-abelian gauge theories

Dániel Nógrádi

Non-abelian gauge theories are well-known and used a lot

QCD

Fermion content unusual $\rightarrow$ more than just a UV fixed point

(Physics Beyond Standard Model $\rightarrow$ we might need unusual fermion content)

IR fixed points, more than one UV fixed point, fixed point merger, ...

Non-abelian gauge theory in $D = 4$ dimensions

Let's consider $SU(N)$ gauge theory on $\mathbb{R}^4$ coupled to N_f flavors of fermions in representation R

$$S = -\frac{1}{4g_0^2}\int d^4x \,\mathrm{Tr}\, F_{\mu\nu}F_{\mu\nu} + \int d^4x \sum_f \bar{\psi}_f (D + m_f)\psi_f$$

Gauge action + fermion action

g_0: bare coupling

m_f: bare fermion masses

Non-abelian gauge theory in $D = 4$ dimensions

Gauge sector, $S_g = -\frac{1}{4g_0^2} \int d^4x \mathrm{Tr}\, F_{\mu\nu} F_{\mu\nu}$

$A_\mu(x)$: $N \times N$ anti-hermitian, traceless, bosonic fields

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

Gauge transformation: $g(x) \in SU(N)$

$$A_\mu \to g A_\mu g^{-1} - \partial_\mu g \cdot g^{-1}$$

$$F_{\mu\nu} \to g F_{\mu\nu} g^{-1}$$

$\mathrm{Tr}\, F_{\mu\nu} F_{\mu\nu} =$ invariant

Non-abelian gauge theory in $D = 4$ dimensions

Fermion sector, $S_f = \int d^4x \sum_f \bar{\psi}_f (D + m_f) \psi_f$

$D = \gamma_\mu (\partial_\mu + A_\mu)$ Dirac operator

above A_μ acts in representation R of $SU(N)$

ψ_f fermion field

$f = 1 \ldots N_f$ number of flavors (or copies)

S_f also gauge invariant

Non-abelian gauge theory in $D = 4$ dimensions

Often we are interested in $m_f = 0$.

Classically S scale invariant g_0 dimensionless

Classically chirally invariant $\psi \to \psi + i\varepsilon\gamma_5\psi$

Non-abelian gauge theory in $D = 4$ dimensions

In QFT, formally:

$$\langle \mathcal{O}(A_\mu, \psi) \rangle = \frac{\int DAD\psi D\bar{\psi}\ \mathcal{O}(A_\mu, \psi)\ e^{-S}}{\int DAD\psi D\bar{\psi}\ e^{-S}}$$

There are divergences $\rightarrow$ regularization $+$ renormalization

If $m_f = 0$ we only have 1 dimensionless coupling: $g_0 \rightarrow g_R(\mu)$

Non-abelian gauge theory in $D = 4$ dimensions

Everything defined by N, N_f, R.

Questions:

- What about other terms in the action?
- What phases and fixed points are there?
- What is the renormalization group trajectory $g^2(\mu)$?
- Relevant/irrelevant operators?
- How does all this depend on N, N_f, R?

Non-abelian gauge theory in $D = 4$ dimensions

Typical example: QCD: $N = 3, N_f = 2 + 1 + 1 + 1 + 1, R = fund$

1-loop perturbation theory:

$$\mu \frac{dg}{d\mu} = \beta(g) = \beta_1 \frac{g^3}{16\pi^2}$$

$$\beta_1 = -\frac{11}{3} N + \frac{2}{3} N_f$$

Asymptotic freedom: $\beta_1 < 0, N_f < \frac{11N}{2}$

This is true with $N = 3, N_f = 6$

QCD

Fixed points?

$\beta(g_*) = 0 \rightarrow g_* = 0$ Gaussian UV fixed point

Perturbation theory trustworthy around $g_* = 0$

Operators classified as relevant/irrelevant by perturbation theory

Essentially power counting, our action contains all relevant operators

QCD with large N_f

If $\beta_1 > 0$ i.e. $N_f > \frac{11N}{2}$ not asymptotically free

Trivial theory like ϕ^4 in $D = 4$

QCD

$g(\mu)$ asymptotically free coupling $g(\mu) \sim 1/\log(\mu/\Lambda)$

Classical action scale invariant, Λ is dynamically generated

Hadron masses $\sim \Lambda$

Scale invariance broken

Also $\bar{\psi}\psi \sim \Lambda^3 \neq 0$

Chiral symmetry also broken

Non-abelian gauge theory in $D = 4$ dimensions

Are the above properties true for all asymptotically free N, N_f, R?

Non-abelian gauge theory in $D = 4$ dimensions

$$\beta_1 = -\frac{11}{3}N + \frac{4}{3}T(R)N_f$$

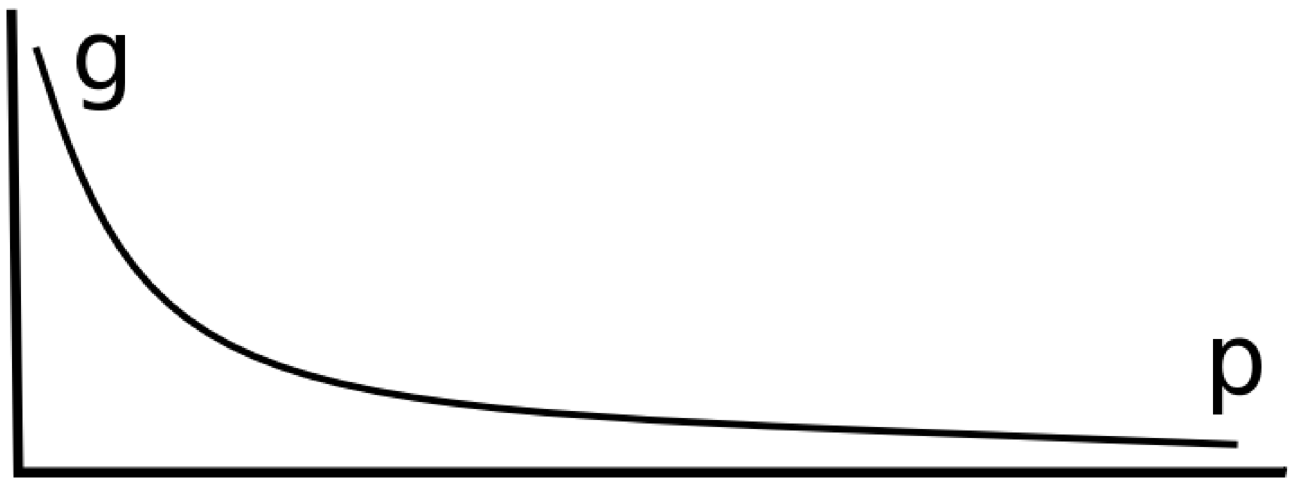
Here $T(R)$ Dynkin index or trace normalization factor of rep R

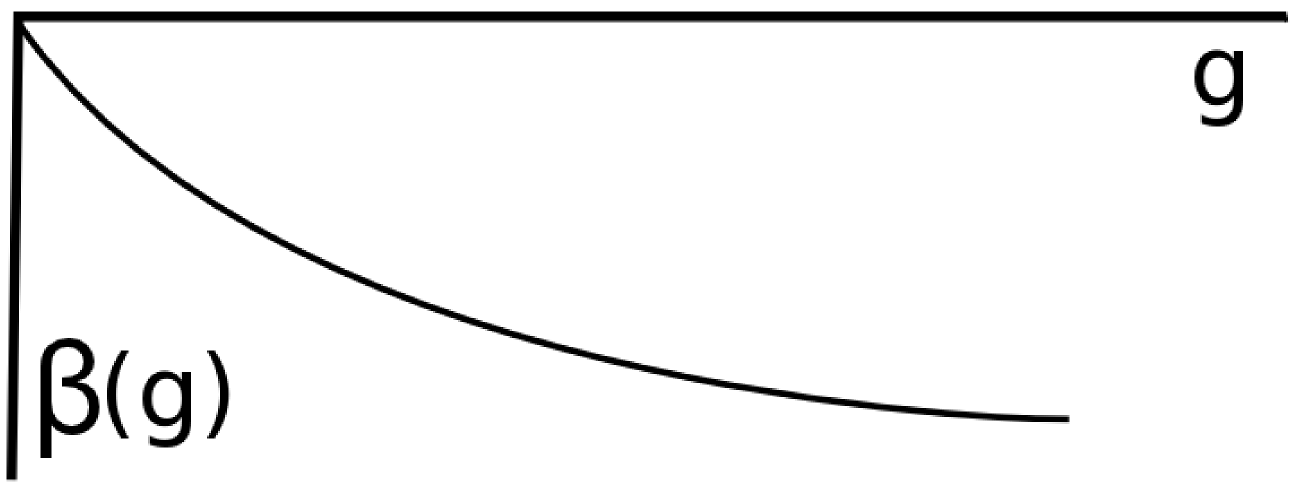
So fix $N_f < \frac{11N}{4T(R)}$

Properties if QCD-like:

- $g_* = 0$ only fixed point, UV
- Scale invariance broken
- Chiral symmetry broken

QCD





Different features we might find:

- $g_* \neq 0$ UV fixed point
- $g_* \neq 0$ IR fixed point
- Scale invariance on the quantum level, 4D CFT
- Chirally symmetric phase on the quantum level
- Different classification of operators as relevant/irrelevant, depending on the fixed point

N_f-dependence

Let's fix N and R

Looking for fixed points on the 2-loop level

$$
\begin{aligned}
\beta(g) &= \mu\frac{dg}{d\mu} = \beta_1\frac{g^3}{16\pi^2} + \beta_2\frac{g^5}{(16\pi^2)^2} \\
\beta_1 &= -\frac{11}{3}N + \frac{4}{3}N_f T(R) \\
\beta_2 &= -\frac{34}{3}N^2 + \left(\frac{5}{3}N + C_2(R)\right) 4T(R)N_f
\end{aligned}
$$

$g_* = 0$ is still UV fixed point

N_f-dependence

Non-trivial fixed point $\beta(g_*) = 0$:

Exists if $\beta_1 < 0$ and $\beta_2 > 0$

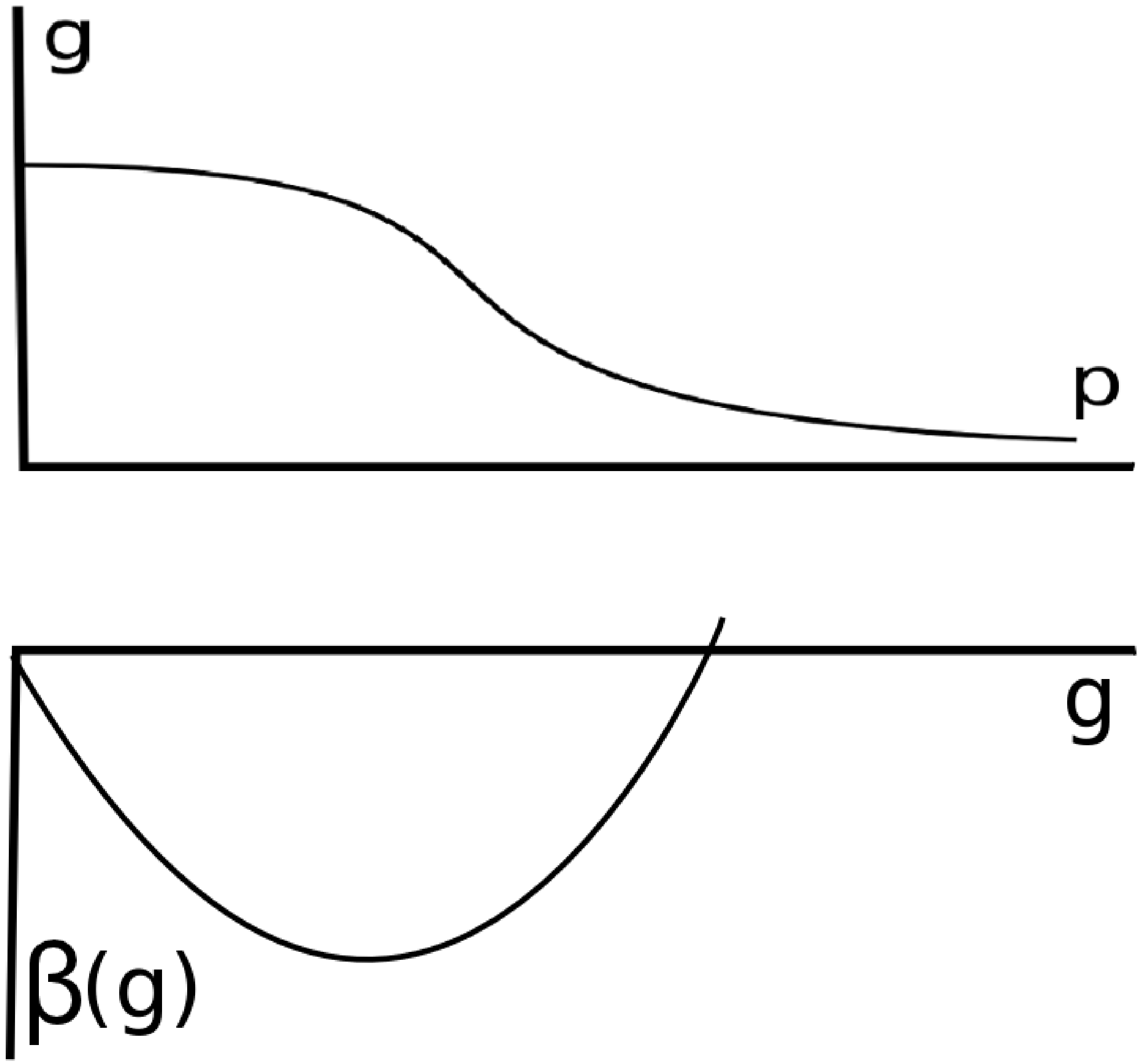
$$g_* = 4\pi\sqrt{-\frac{\beta_1}{\beta_2}}$$

$$N_f^{low} = \frac{34N^2}{4T(R)(5N+3C_2(R))} < N_f < \frac{11N}{4T(R)} = N_f^{up}$$

This N_f range is the conformal window

Fixed point g_* an IR fixed point.

If there is an IR fixed point



N_f-dependence

How trustworthy is this?

$$N_f^{low} = \frac{34N^2}{4T(R)(5N+3C_2(R))} < N_f < \frac{11N}{4T(R)} = N_f^{up}$$

Upper end of the conformal window: loss of asymptotic freedom
$\longrightarrow$ perturbation theory is trustworthy, even 1-loop is enough

$g_* = 4\pi\sqrt{-\frac{\beta_1}{\beta_2}}$ is small because β_1 is small

Lower end of the conformal window: 2-loop is suspect

$g_* = 4\pi\sqrt{-\frac{\beta_1}{\beta_2}}$ is large because β_2 is small

N_f-dependence

Where we know what we are doing: close to upper end of the conformal window

E.g. $N = 3$, $R = fund$, $N_f^{up} = 16.5$

For example $N_f = 16$ 2-loop result is probably okay, a non-trivial weakly interacting 4D CFT

N_f-dependence

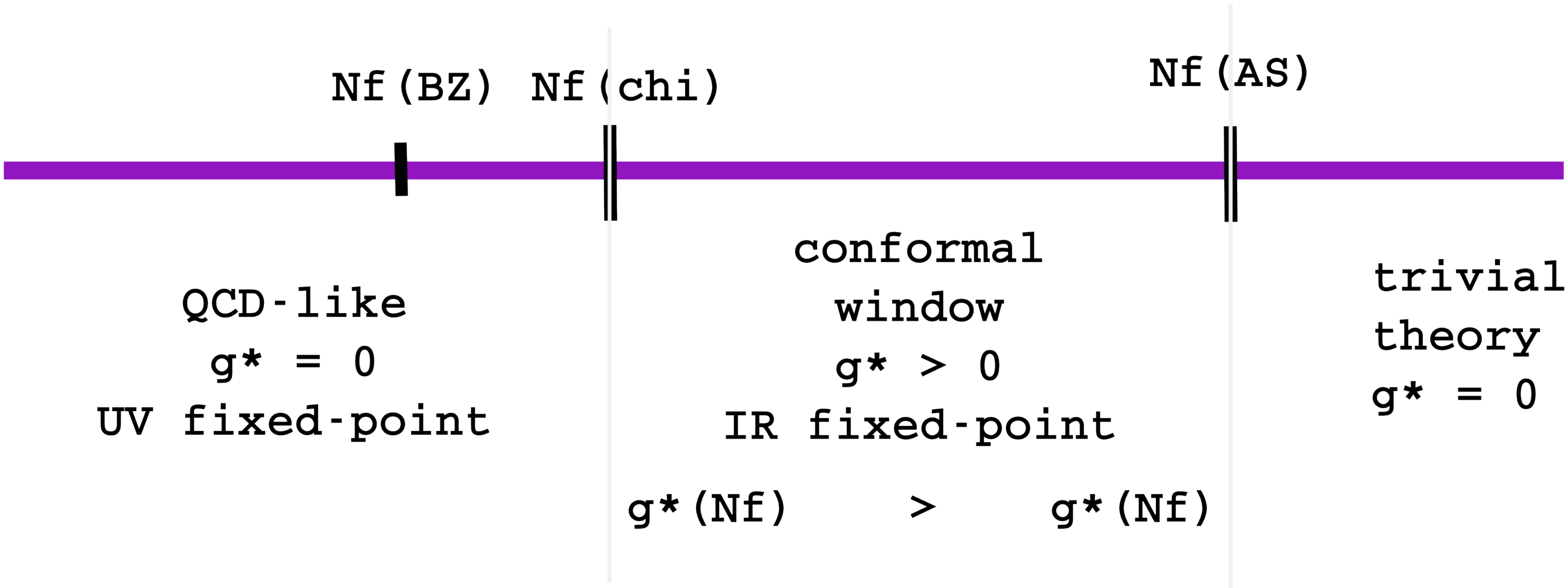
Even though 2-loop result is unreliable for N_f^{low} the lesson is that there exists an N_f^{low} but we can't compute it in perturbation theory

Is real N_f^{low} smaller or larger than 2-loop N_f^{low}?

Probably larger.

As N_f decreases from upper end of conformal window g_* grows $\longrightarrow$ if not too large still CFT $\longrightarrow$ as it gets large chiral symmetry breaks $\longrightarrow$ scale is generated $\longrightarrow$ conformal symmetry lost $\longrightarrow$ no IR fixed point $\longrightarrow$ we are outside the conformal window.

N_f-dependence summary



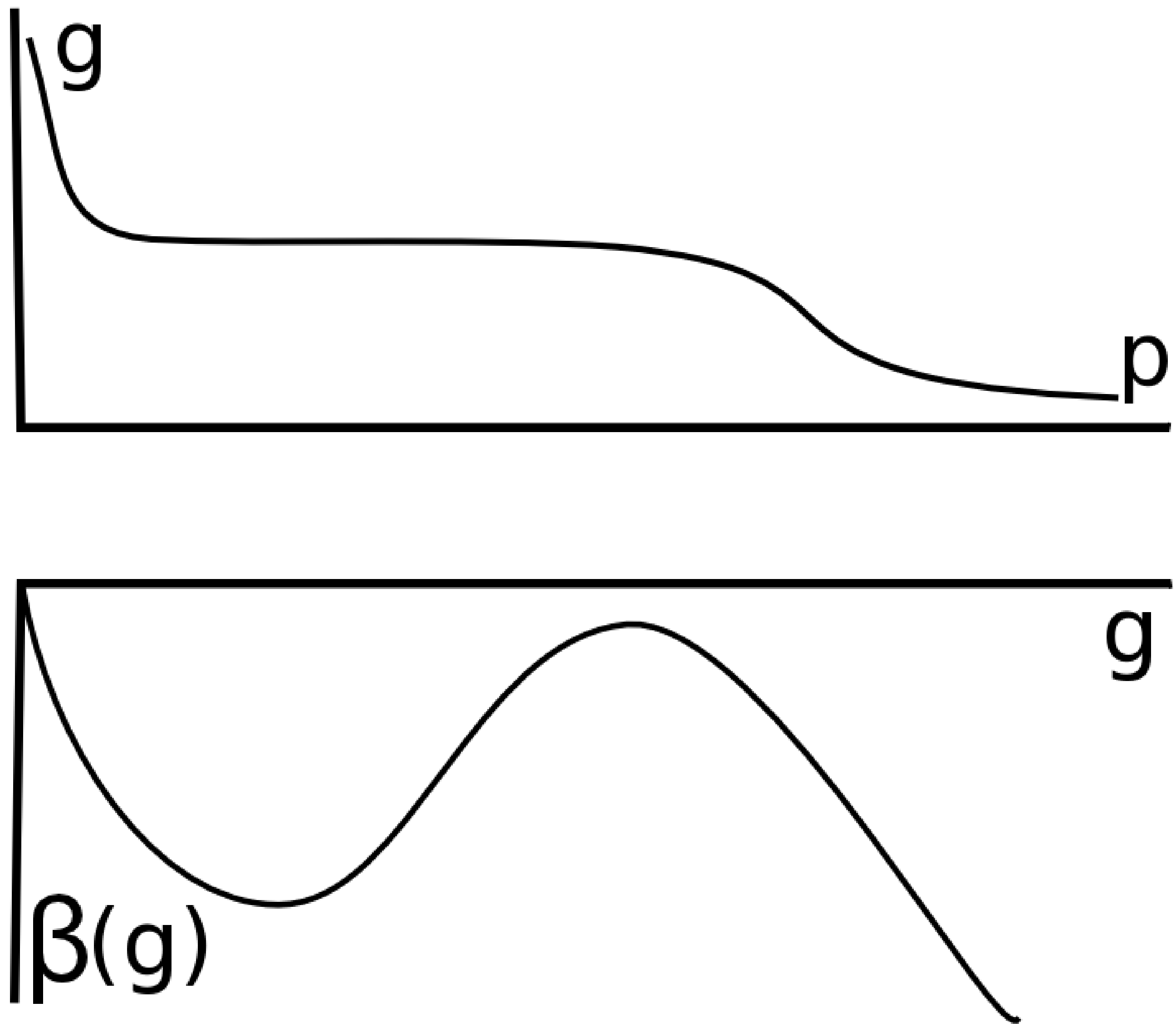
N_f increases from left to right

Note on $N_f = 1$

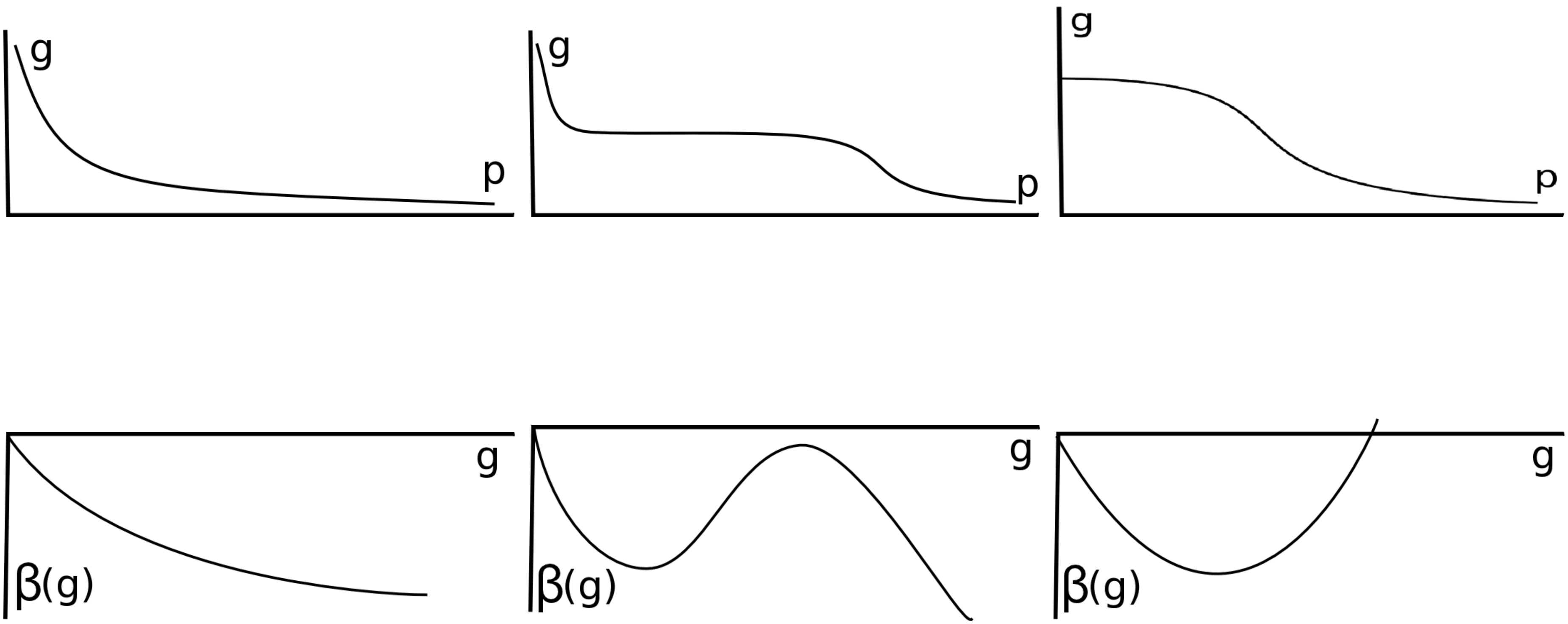
$N_f = 1$ theory special

Chiral anomaly $\rightarrow$ $SU(N_f) \times SU(N_f)$ chiral group completely broken by anomaly $\rightarrow$ no spontaneous breaking $\rightarrow$ no Goldstone pions $\rightarrow$ every mass $\sim \Lambda$

N_f just below lower end of conformal window



N_f-dependence 2. summary



N_f increases from left to right

N_f just above lower end of conformal window

Expect 3 fixed points!

1. $g_*^{UV1} = 0$ usual UV fixed point

2. $g_*^{IR} \neq 0$ IR fixed point

3. $g_*^{UV2} \neq 0$ new UV fixed point

$$g_*^{UV1} < g_*^{IR} < g_*^{UV2}$$

Unfortunately g_*^{IR} and g_*^{UV2} non-perturbative.

Question to audience

Is there a model in which the β-function has 3 fixed points and all of them are perturbative i.e. all 3 g_* are small?

Conformal window

N_f^{low} is non-perturbative

Lattice!

We know a bit about g_*^{IR} non-perturbatively in some models

We know nothing about g_*^{UV2} unfortunately

Examples

Perturbative 2-loop N_f^{low}

$SU(2)$

- $R : j = 1/2, \quad 5.551... < N_f < 11$
- $R : j = 1, \quad 1.0625 < N_f < 2.75$
- $R : j = 3/2, \quad 0.32 < N_f < 1.1$

Examples

Perturbative 2-loop N_f^{low}

$SU(3)$

- $R = fund, \quad 8.05... < N_f < 16.5$
- $R = sextet, \quad 1.224 < N_f < 3.3$
- $R = adj, \quad 1.0625 < N_f < 2.75$

For example $R = sextet$ with $N_f = 3$ probably also weakly coupled non-trivial 4D CFT

Inside conformal window

Dynamics very different from outside conformal window

Outside: QCD-like

Inside: CFT

CFT

Gauge coupling irrelevant

Mass term for fermions still relevant

Mass anomalous dimension: γ constant

Dimensionful quantities are zero if $m = 0$ fermion mass

If $m \neq 0$ and $\bar{\psi}\psi$ only relevant operator $\rightarrow$ quantities scale with γ as a function of m

In 1-loop perturbation theory: $\gamma = 6C_2(R)\frac{g_*^2}{16\pi^2}$

Scaling with fermion mass

$$m_H(m) \sim m^{\frac{1}{1+\gamma}} + \ldots$$

$$\sigma(m) \sim m^{\frac{1}{1+\gamma}} + \ldots$$

$$F(m) \sim m^{\frac{1}{1+\gamma}} + \ldots$$

...

Basically dimensional analysis.

CFT

But! We don't know whether there are other relevant operators around $g_* \neq 0$ IR fixed point!

There could be if γ large

Again perturbation theory is reliable close to upper end of conformal window so we start from there and decrease N_f towards lower end of conformal window and even though it will be unreliable it will give useful hints

$(\bar{\psi}\psi)^2$ 4-fermi operator

Irrelevant, dimension 6, around Gaussian UV fixed point

Since $\bar{\psi}\psi$ is dimension $3 - \gamma$, the dimension of 4-fermi operator close to $6 - 2\gamma$ in perturbation theory

If γ close to 1 $\rightarrow$ 4-fermi operator can become relevant!

If perturbation theory reliable (g_* is small) γ is small, $6 - 2\gamma$ still larger than 4

Again perturbative calculation is just a guide, unreliable where we need it

CFT

Example: lattice studies indicate this model has IR fixed point

$SU(2), N_f = 2, R = adj$

$\gamma \sim 0.3$

Smallish coupling, smallish γ, $(\bar{\psi}\psi)^2$ probably irrelevant

Almost CFT

Example: lattice studies indicate this model is just below conformal window

$SU(3), N_f = 2, R = sextet$

Running is slow, but no fixed point

Lattice

What we try to do:

Determine N_f^{low} non-perturbatively

Measure non-perturbative $\beta(g)$

Measure dependence on m fermion mass

Look at finite T transitions

Lattice methods 1

Measure running coupling $g(\mu)$ or $\beta(g)$

On lattice: a finite, L finite

Need: $1/L \ll \mu \ll 1/a$ separation of 3 scales

Easier: $1/L = \mu \ll 1/a$ only separate 2 scales

Step scaling, running with $\mu = 1/L$

Finite scale change $L \to sL$, where $s = 3/2$ or $s = 2$ etc.

$g^2(sL) - g^2(L)$ as a function of $g^2(L)$

Discrete β-function has a zero $\to$ IR fixed point

Scaling with m fermion mass

QCD-like: chiral perturbation theory gives small m-dependence, dictated by pion dynamics, at low energies pions are dominant degrees of freedom $\rightarrow$ chiral logs $+$ analytical terms

CFT: scaling with γ

Need to measure mass spectrum, decay constants, etc,

Need: $1/L \ll M(m) \ll 1/a$

Difficult, because really need 3 separate scales

Or can incorporate finite L behavior

Lattice methods 2

Finite L behavior

QCD-like: finite volume chiral perturbation theory

CFT: finite size scaling, $x = Lm^{\frac{1}{1+\gamma}}$ scaling variable, many different volumes and masses fall on the same universal curve $f(x)$

Finite T transitions

QCD-like: $T = 0$ chirally broken, $T \gg 0$ chirally symmetric $\rightarrow$ transition at $T = T_c$

CFT: $T = 0$ scale and chirally symmetric, $T \gg 0$ chirally symmetric $\rightarrow$ all T same phase, no T_c

Difficulty: lattice discretized system has fake transitions which are lattice artifacts, specific to discretization, non-universal

Lattice issues

In order to see IR behavior m: small, L: large

L large: expensive obviously

m small: also expensive, because we need to invert $D+m$ and condition number of $D+m$ is proportional to $1/m$

The larger the fermion content (either large N_f or large dimension for R) the more expensive the computation is

Over-all: much more expensive than QCD

Further problems: in interesting models, coupling runs slowly

$g^2(sL) - g^2(L)$: here each term is $O(1)$, difference is small $\longrightarrow$ large cancellation $\longrightarrow$ need very small errors

Lattice issues

Major advance in QCD studies: improvements!

Improved gauge action, improved fermion action

Gets us closer to continuum limit by $O(a^n)$, $n = 2, 3$ etc.

Based on perturbative calculation around UV fixed point

If IR fixed point $\rightarrow$ we don't know whether they help or not. Can make scaling even worse!

Conceptual issues

Transition as N_f changes $\longrightarrow$ we pretend N_f is continuous $\longrightarrow$ it is not

Only (half)integer N_f are meaningful theories

Maybe intuition from continuous N_f misleading?

Conceptual issues

Maybe there are unexpected relevant operators?

Should be detectable by lack of scaling ...

Summary and conclusion

Tried to explain what type of models we are working on

Didn't explain why .... Beyond Standard Model, composite Higgs particle

Tried to explain the context and main challenges

Cond-mat people have much more experience with these kinds of phenomena, if perhaps not in the same models

Hopefully I can learn something from stat-phys, cond-mat crowd.

Thank you for your attention!