Strong Griffiths singularities in random systems and their relation to extreme value statistics

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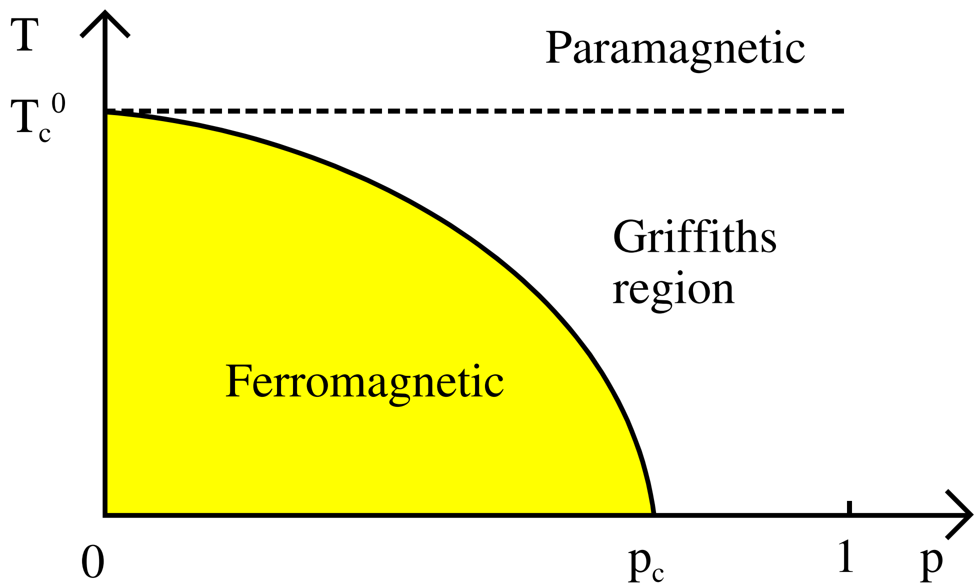
AGENDA

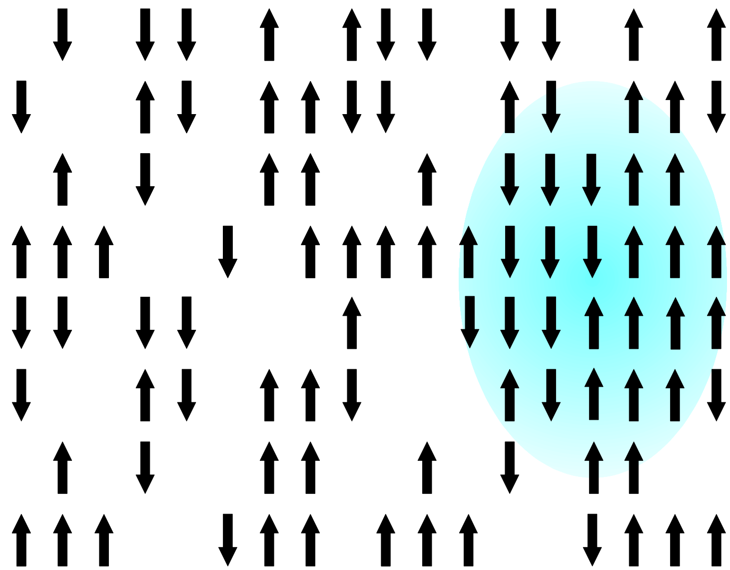
- Introduction - rare region effects
- Griffiths singularities in random quantum systems
  - Random transverse Ising model with extreme disorder
  - Strong disorder RG method
  - Scaling results
  - Numerical tests
- Griffiths singularities in random stochastic systems
  - Partially asymmetric simple exclusion process with extreme disorder
  - Strong disorder RG and scaling results
  - Numerical tests
- Conclusion

Rare region effects

Randomly diluted classical Ising model

$$H = -J \sum_{\langle ij \rangle} \kappa_i \kappa_j S_i S_j + h \sum_i \kappa_i S_i, \qquad P(\kappa) = p\delta(\kappa) + (1-p)\delta(1-\kappa)$$





Rare region of size l - locally in the ferromagnetic phase

density $\sim (1-p)^{l^d}$

relaxation time $\tau \sim \exp(Al^{d-1})$

autocorrelation: $\ln G(t) \sim -(\ln t)^{d/(d-1)}$

magnetization: $m(h) \sim \exp(-B/h)$

weak Griffiths singularities

Random transverse-field Ising model (RTIM)

$$H = -\sum_{i=1}^{L-1} \lambda_i \sigma_i^x \sigma_{i+1}^x - \sum_{i=1}^{L} h_i \sigma_i^z \ ,$$

$\sigma_i^{x,z}$: Pauli matrices
λ_i: couplings, h_i: transverse fields, iid random numbers.
control parameter

$$\delta = \frac{[\ln h]_{\rm av} - [\ln \lambda]_{\rm av}}{{\rm var}[\ln \lambda] + {\rm var}[\ln h]} \ .$$

$T = 0$, $\delta < 0$: ferromagnetic phase; $\delta > 0$: paramagnetic phase.
$\delta = 0$: random quantum critical point
time-scale is governed by the largest coupled region
$\tau \sim 1/\epsilon \sim L^z$, $z = z(\delta)$ dynamical exponent
autocorrelation: $G(t) \sim t^{-1/z}$
susceptibility: $\chi \sim T^{-1+1/z}$

strong Griffiths singularities

Partially asymmetric simple exclusion process (PASEP)

- N particles hop to neighboring empty sites of a $1d$ lattice of size $L > N$
- the hop rates could depend on
  - the given particle (particle-wise (pw) disorder)
  - or on the departure site (site-wise (sw) disorder)
- the hop rates for the i-the particle (site): forward p_i, backward q_i

control parameter:

$$\delta_p = \frac{[\ln p]_{\rm av} - [\ln q]_{\rm av}}{\mathrm{var}[\ln p] + \mathrm{var}[\ln q]} ,$$

the particles move to the right (to the left) for $\delta_p > 0$ ($\delta_p < 0$).
time-scale is governed by the largest barrier
Stationary velocity: $v \sim 1/\tau \sim L^{-z_p}$
$z_p = z_p(\delta_p)$: dynamical exponent

strong Griffiths singularities

Extreme value statistics (EVS)

- $y_1, y_2, \ldots, y_L$ (independent) random numbers
- distributed with (identical) parent distribution $\pi(y)$
- question is the distribution of the largest (k-th largest) value: y_{max}.

For iid random numbers three basic universality classes, depending on $\lim_{y\to\infty} \pi(y)$.

- $\pi(y)$ decays faster than any power-law: Gumbel distribution.
- $\pi(y)$ decays as a power-law: Fréchet distribution.
- $\pi(y)$ has a power-law with an edge: Weibull distribution.

For non-iid random numbers no general results.

Strong Griffiths singularities are governed by rare, extreme regions.

The interacting many-particle systems have strong correlations.

Can the EVS still be of relevance?

Exact result: RTIM with extreme disorder

Bimodal distribution: $h_i = 1, \quad \lambda_i = \begin{cases} \lambda & \text{with pr} \quad c \\ \lambda^{-1} & \text{with pr} \quad 1-c \end{cases}$

extreme limit: $c \ll 1$; $\lambda \gg 1 \quad \rightarrow \quad \delta \sim (1-2c)\ln\lambda \gg 1$ paramagnetic phase.

Properties of a **rare region** of n strong bonds:

density of the cluster: $\rho(n) = c^n$

excitation energy: $\epsilon(n) \approx \lambda^{-n} \quad \rightarrow \quad n = -\frac{\ln\epsilon}{\ln\lambda}, \quad \mathrm{d}n = -\frac{\mathrm{d}\epsilon}{\epsilon\ln\lambda}$

distribution of the low-energy excitations:

$$P(\epsilon)\mathrm{d}\epsilon = \rho(n)\mathrm{d}n \quad \rightarrow \boxed{P(\epsilon) \approx \frac{1}{\ln\lambda}\epsilon^{\omega}, \quad \epsilon \rightarrow 0}; \quad \omega = \frac{\ln(1/c)}{\ln\lambda} - 1$$

Typical size of the largest cluster: $n_1 \quad \rightarrow \quad L\sum_{n\geq n_1}\rho(n) = 1 \quad \rightarrow \quad n_1 \approx \frac{\ln L}{\ln(1/c)}$

smallest gap: $\boxed{\epsilon_1 \approx \lambda^{-n_1} \sim L^{-z}}$

with the dynamical exponent: $z = \frac{\ln\lambda}{\ln(1/c)}$, and $\omega = \frac{1}{z} - 1$.

Distribution of the smallest gaps

$$P_L(\epsilon_1) = L^z \tilde{P}_1(\epsilon_1 L^z) \sim L\epsilon_1^{\omega}$$

- localized excitations: the largest cluster can be at $\sim L$ positions
- ϵ_1 is the smallest gap out of $\sim L$ **independent** rare regions,
- having **identical** parent **distribution**.

$\tilde{P}_1(u)$, is the standard Fréchet distribution

$$\boxed{\tilde{P}_1(u) = \frac{1}{z} u^{1/z-1} \exp(-u^{1/z})}$$

for the k-th smallest gap:

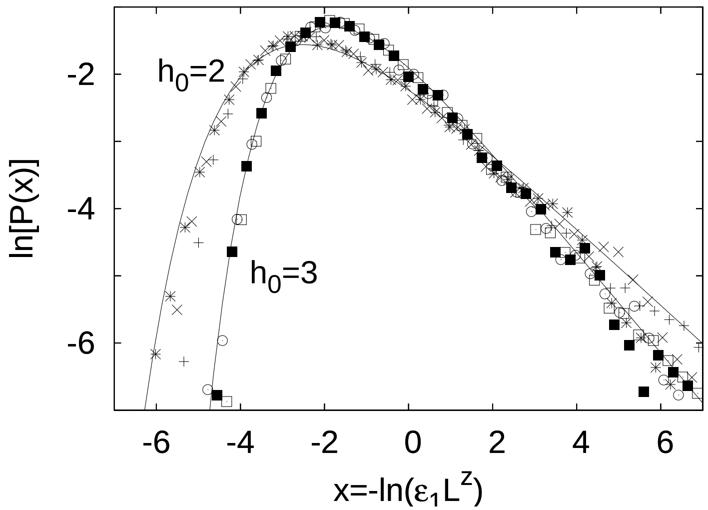
$$\tilde{P}_k(u_k) = \frac{1}{z} u_k^{k/z-1} \exp(-u_k^{1/z}), \quad u_k = u_0 L^z \epsilon_k$$

Numerical test

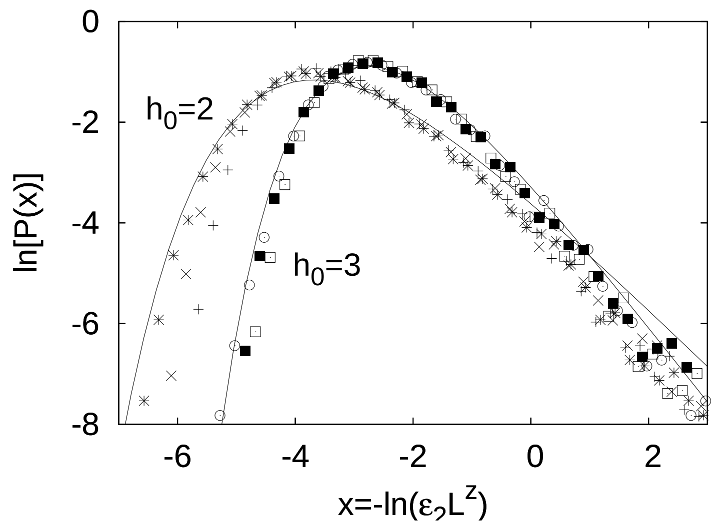
uniform distribution:

$$\pi_\lambda(\lambda) = \begin{cases} 1 & \text{for} \quad 0 < \lambda < 1 \\ 0 & \text{otherwise} \end{cases} \qquad \pi_h(h) = \begin{cases} 1/h_0 & \text{for} \quad 0 < h < h_0 \\ 0 & \text{otherwise} \end{cases}$$

dynamical exponent $z \quad \rightarrow \quad z \ln(1 - z^{-2}) = -\ln h_0.$



RTIM first gap



RTIM second gap

Exact result: PASEP with extreme disorder

Particle-wise bimodal disorder:
black particles: a fraction of c, $p_i = 1, \; q_i = \lambda$
white particles: a fraction of $1 - c$, $p_i = 1, \; q_i = \lambda^{-1}$
extreme limit: $c \ll 1$; $\lambda \gg 1 \quad \to \quad \delta \sim (1 - 2c) \ln \lambda \gg 1$ drift to the right.
Properties of a **rare region** of a cluster of n black particles:

density of the cluster: $\rho(n) = c^n$

speed of the cluster: $v(n) \approx \lambda^{-n}$

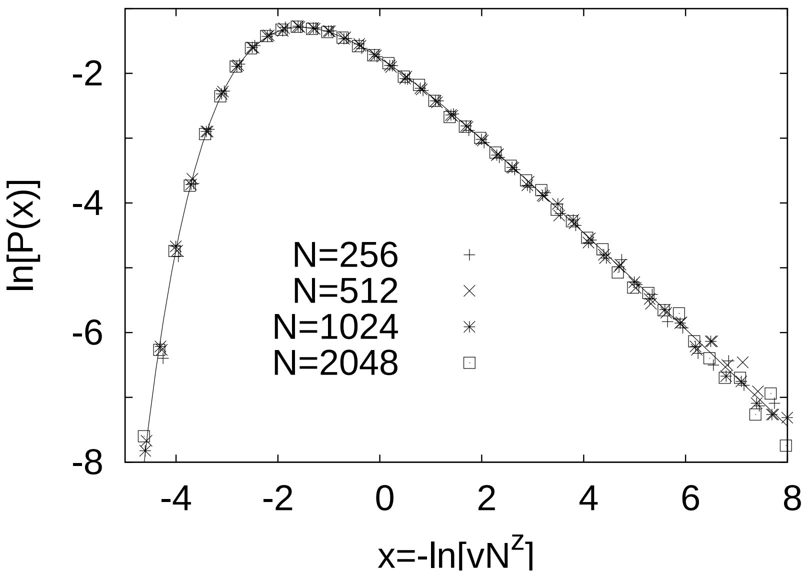
distribution of the speed of clusters:

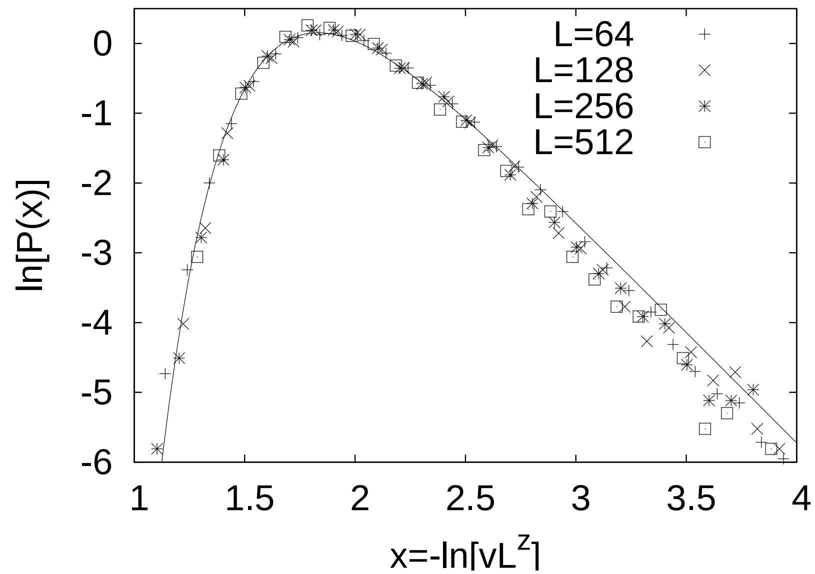
$$\boxed{P(v) \approx \frac{1}{\ln \lambda} v^{\omega}, \quad v \to 0}; \qquad \omega = \frac{\ln(1/c)}{\ln \lambda} - 1$$

Equivalence with the RTIM as

$$\boxed{\epsilon(n) \leftrightarrow v(n)}$$
$$\boxed{\epsilon_1 \leftrightarrow v_1 \equiv v_{stationary}}$$

Numerical test





pw uniform disorder: $h_0 = 3.$

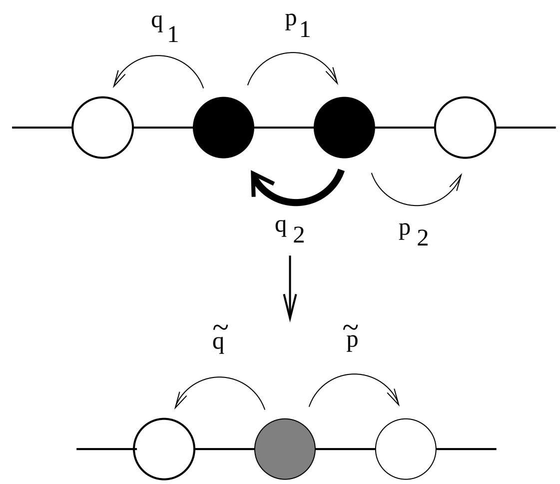
sw binary disorder: $c = 0.25, \ \lambda = 2$

$z_{sw} = \frac{z_{pw}}{2}$

Strong disorder renormalization

F.I., C. Monthus, Physics Reports **412**, 277-431, (2005)

1. PASEP Composite particle formation



Renormalization scheme for particle clusters. If q_2 is the largest hopping rate, in a time-scale, $\tau > 1/q_2$, the two-particle cluster moves coherently and the composite particle is characterized by the effective hopping rates $\tilde{q}$ and $\tilde{p}$, respectively.

- largest hop rate defines the energy scale: $\Omega = q_2 \gg p_1, q_1, p_2$
- time scale: $\tau = 1/\Omega$, consider only $t > \tau$:
- $\tilde{q} = q_1 \times \frac{q_2}{q_2 + p_1} \approx q_1$
- $\tilde{p} = p_2 \times \frac{p_1}{q_2 + p_1} \approx \frac{p_1 p_2}{q_2}$
- for $\Omega = p_1 \gg p_2, q_1, p_2$ we have $\boxed{\tilde{p} \approx p_2,\ \tilde{q} \approx \frac{q_1 q_2}{p_1}}$

Strong disorder RG approach of the random XX-chain

- Hamiltonian:

$$H_{XX} = -\sum_{i=1}^{2L} J_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right)$$

- largest coupling defines the energy scale: $\Omega = J_2 \gg J_1, J_3$

- two sites with J_2 form an effective singlet - are decimated out

- effective coupling between remaining sites: $\tilde{J} \approx \frac{J_1 J_3}{J_2}$

- Correspondence with the PASEP: $q_i \leftrightarrow J_{2i-1} \quad p_i \leftrightarrow J_{2i}$

2. RG equations for the distribution functions: $P(p,\Omega), R(q,\Omega)$

$$\begin{aligned}
\frac{\mathrm{d}R(q,\Omega)}{\mathrm{d}\Omega} &= R(q,\Omega)[P(\Omega,\Omega)-R(\Omega,\Omega)] \\
&- P(\Omega,\Omega)\int_q^\Omega \mathrm{d}q' R(q',\Omega)R(\frac{q\Omega}{q'},\Omega)\frac{\Omega}{q'} \\
\frac{\mathrm{d}P(p,\Omega)}{\mathrm{d}\Omega} &= P(p,\Omega)[R(\Omega,\Omega)-P(\Omega,\Omega)] \\
&- R(\Omega,\Omega)\int_p^\Omega \mathrm{d}p' P(p',\Omega)P(\frac{p\Omega}{p'},\Omega)\frac{\Omega}{q'},
\end{aligned}$$

3. Fixed-point solution at $\Omega = \Omega^* \to 0$

The asymmetric model, $\delta > 0$

$$P_0(p, \Omega) \approx \frac{1}{z\Omega} \left(\frac{\Omega}{p}\right)^{1-1/z}, \quad \Omega < \Omega_\xi \sim \xi^{-z}$$

z: dynamical exponent is the solution of:

$$\left[\left(\frac{q}{p}\right)^{1/z}\right]_{\mathrm{av}} = 1$$

At an energy-scale, $\Omega_0 \ll \Omega_\xi$, typically

$$\tilde{p} \sim \Omega_0, \quad \ln \tilde{q} \sim \Omega_0^{-1/z}$$

Renormalized model

- non-interacting effective particles of finite mass $\leftrightarrow$ localized excitations
- unidirectional move with random speeds - identical distribution
- the stationary velocity is given by the smallest speed

Low energy excitations - EVS - Fréchet distribution

Numerical RG tests for random quantum systems

- Finite random quantum system of linear size L
- Numerical renormalization up to the last effective spin (spin singlet)
- Last effective gap, ϵ, is calculated
- Its distribution is plotted
- Gap exponent, ω, is measured from the tail of the distribution

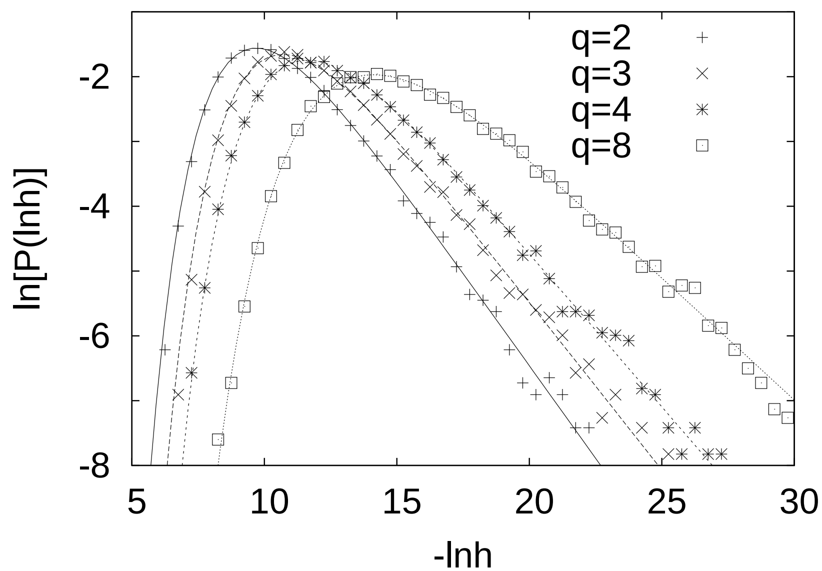
$$P_L(\epsilon) = L^z \tilde{P}_1(\epsilon L^z) \text{ ? } \to \text{? } \sim L^d \epsilon^\omega$$

- Dynamical exponent, z, is calculated from scaling collapse
- Localization of excitations is checked: $\omega + 1 = \frac{d}{z}$
- Comparison is made with the Fréchet distribution

Random quantum Potts chain

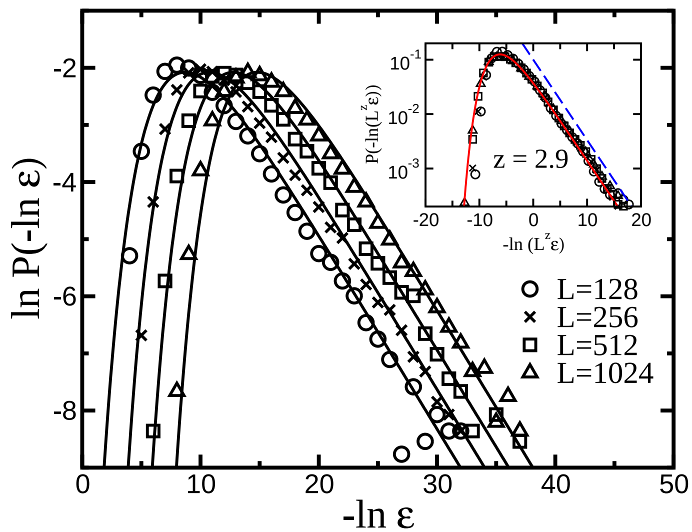
$$H_P = -\sum_{i=1}^{L-1} \lambda_i \delta(s_i, s_{i+1}) - \sum_{i=1}^{L} \frac{h_i}{q} \sum_{k=1}^{q-1} M_i^k$$

q-state spin variables: $|s_i\rangle = |1\rangle = |2\rangle = \ldots |q\rangle$

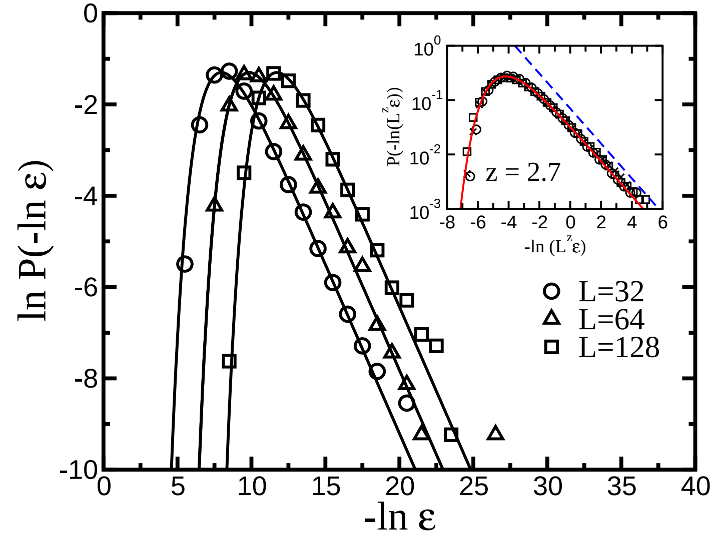


$L = 2048$, uniform disorder: $h_0 = 3$
(EVS seems to work)

RTIM: ladder and $2d$ system



RTIM ladder, uniform disorder:
$h_0 = 2.5$
(EVS seems to work)

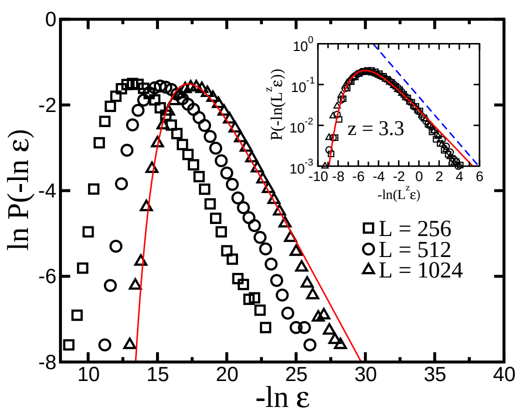


RTIM in $2d$, uniform disorder: $h_0 = 9$.
(EVS seems to work)

Random Heisenberg models

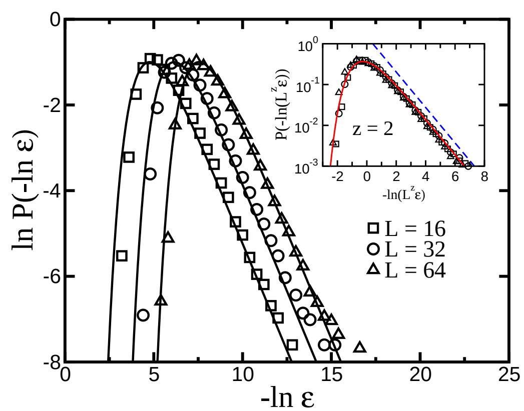
$$H_H = \sum_{i,j} J_{i,j} t_i t_j \vec{S}_i \cdot \vec{S}_j$$

$\vec{S}_i$: spin-1/2 variable, $t_i = 0$ with probability, p, and $t_i = 1$, otherwise.



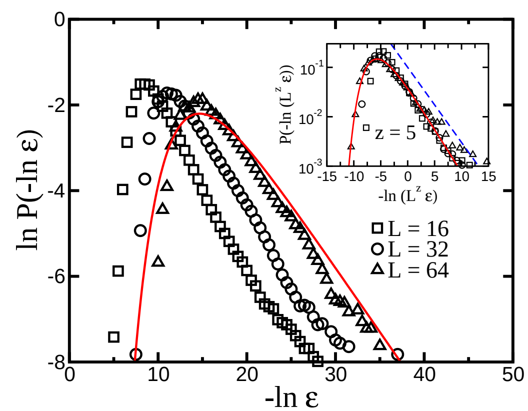
chain with $-0.5 < J_i < 0.5$

(EVS does not work)



square lattice with Gaussian disorder of variance 1

(EVS seems to work)



diluted square lattice ($p = 0.125$) with uniform AF disorder

(EVS does not work)

Random quantum systems

EVS probably works, if the RG has same type of (strong disorder) fixed point.

- Models with discrete symmetry (Ising, Potts, etc)
  - similar decimation rules
  - localized excitations
  - EVS could work at any dimension
- Models with continuous symmetry (Heisenberg)
  - for non-chain.like objects modified decimation rules
  - large spin formation $\rightarrow$ non-localized excitations
  - EVS generally does not work for $d > 1$

Conclusions

- Strong Griffiths singularities are due to rare region effects
- In systems with discrete symmetry the rare regions are localized
- Strong disorder RG provides low energy excitations, which are
  - non-interacting $\rightarrow$ independent
  - identically distributed random variables
  - the low-energy tail is algebraic

Low energy excitations - EVS - Fréchet distribution