INTRODUCTION TO BLACK HOLE THERMODYN. – THE CLASSICAL THEORY

1. Gravitational collapse of a dust ball
2. The geometry of Kerr-Newman black holes
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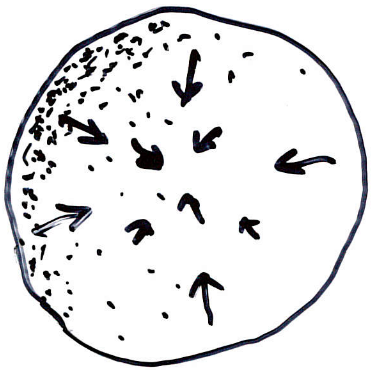
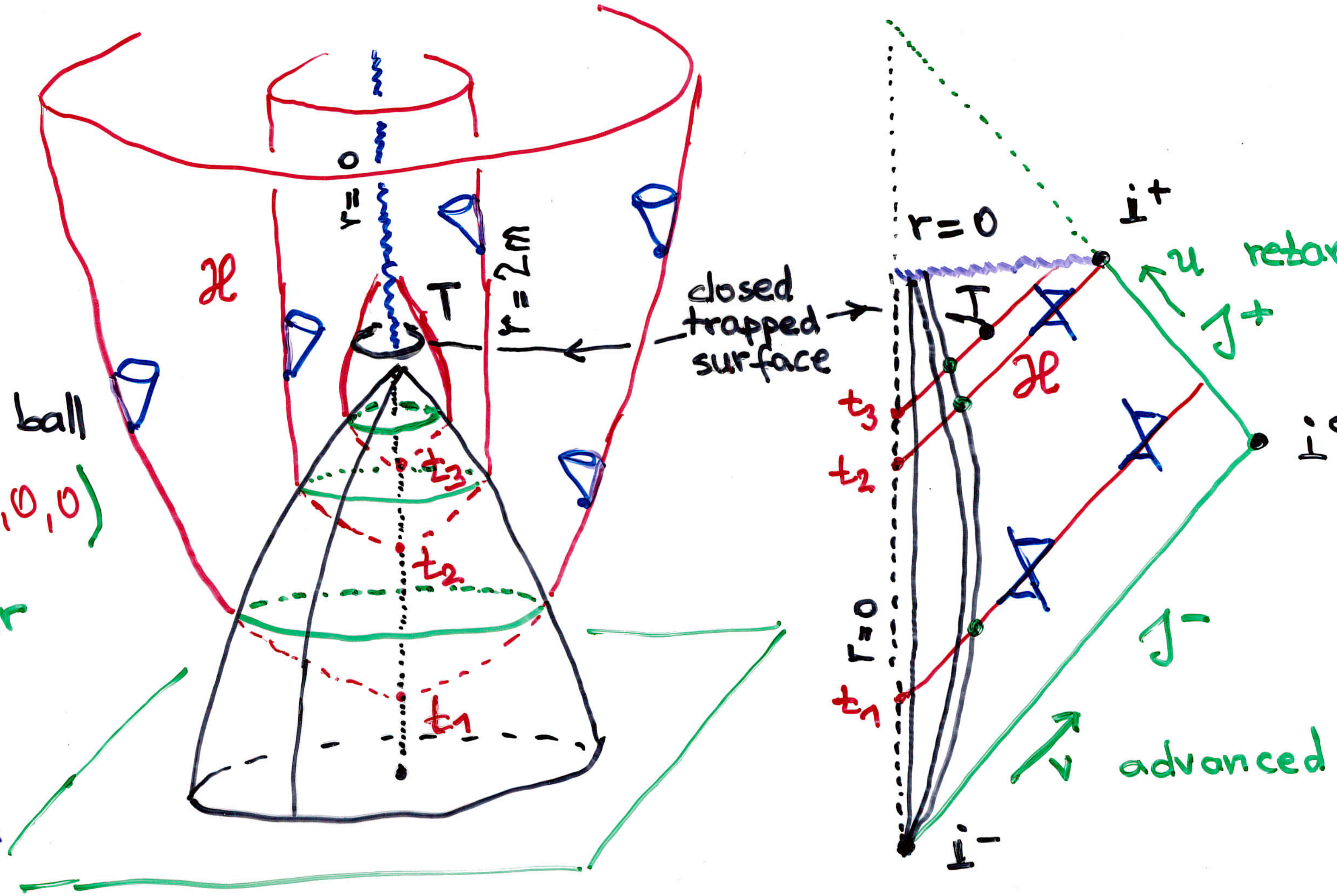
1. Gravitational collapse of a dust ball



spherical dust ball

$T_{ab} = \mathrm{diag}(\mu(r), 0, 0, 0)$

$m := 4\pi \int_0^\infty \mu(r)\, r^2\, dr$



Metric outside:

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{2m}{r}} - r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$$ – Schwarzschild

– static, spherically symmetric black hole final state

2. The geometry of Kerr-Newman black holes

- Kerr-Newman sol. of Einstein-Maxwell eqs.:

$$ds^2 = \frac{\Delta}{R^2}(dt - a\sin^2\theta\, d\phi)^2 - \frac{\sin^2\theta}{R^2}(a\,dt - [r^2+a^2]\,d\phi)^2 - R^2 d\theta^2 - \frac{R^2}{\Delta}dr^2$$

and $\bar{A} = -\frac{e\,r}{R^2}(dt - a\sin^2\theta\, d\phi)$ – 4-vector potential

where

$$\Delta := r^2 - 2mr + a^2 + e^2, \quad R^2 := r^2 + a^2\cos^2\theta \qquad \begin{cases} m & \text{– mass} \\ ma & \text{– ang. mom.} \\ e & \text{– el. charge} \end{cases}$$

Symmetries: $K^a = \left(\frac{\partial}{\partial t}\right)^a$ – stationarity, $X^a = \left(\frac{\partial}{\partial \phi}\right)^a$ – axi-symmetry

- Black hole: when $\exists$ event horizon – at the roots of $\Delta(r) = 0$:

$$r_\pm = m \pm \sqrt{m^2 - a^2 - e^2} \quad \text{or} \quad r_0 = m$$

Criterion of the existence of the root(s): $m^2 \geq a^2 + e^2$.

Extreme K-N black hole: with equality – $m^2 = a^2 + e^2$.

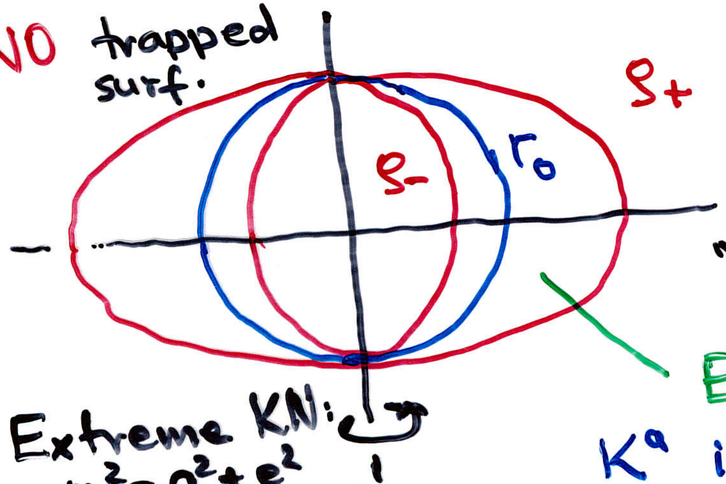
Event horizon ($\mathcal{H}$): null hypersurface, generated by non-expanding null geodesics.

- **Ergosphere:**

$$g_{ab}K^aK^b = \frac{\Delta - a^2\sin^2\theta}{R^2} \begin{cases} \to 1 \text{ for } r\to\infty \\ \text{may be negative!} \end{cases}$$

in fact: $K_aK^a = 0$ at $\varrho_\pm = m \pm \sqrt{m^2 - e^2 - a^2\cos^2\theta}$

NO trapped surf.

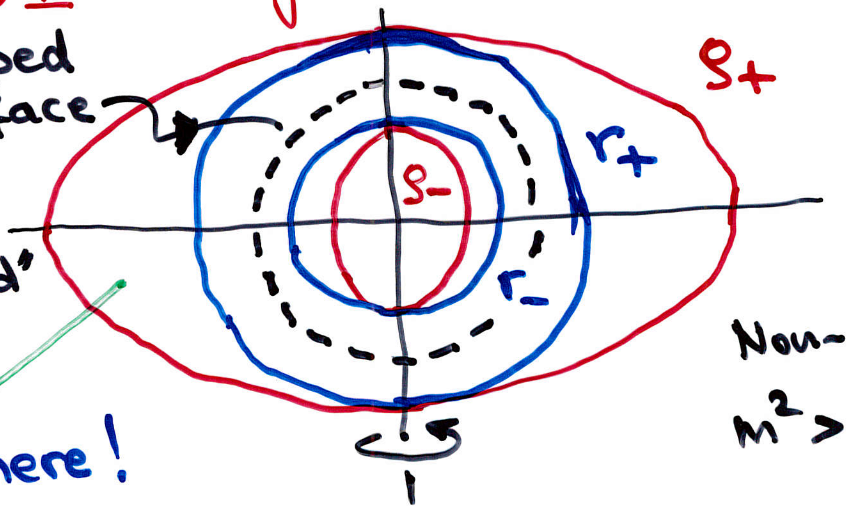


Extreme KN: $m^2 = a^2 + e^2$

$\theta = \frac{\pi}{2}$ 2plane
"side view in 3d"

Ergosphere
K^a is spacelike here!

closed trapped surface



Non-extreme KN
$m^2 > a^2 + e^2$

- The angular velocity: both K^a, X^a are spacelike on $\mathcal{H}$ if $a \neq$

But: $\xi^a = K^a + \underbrace{\frac{a}{r_+^2 + a^2}}_{\Omega_H} X^a$ is timelike outside $\mathcal{H}$ and is null on $\mathcal{H}$.

Ω_H - angular velocity of BH.

Meaning of Ω_H?

For observers being at rest w.r.t. t, i.e.

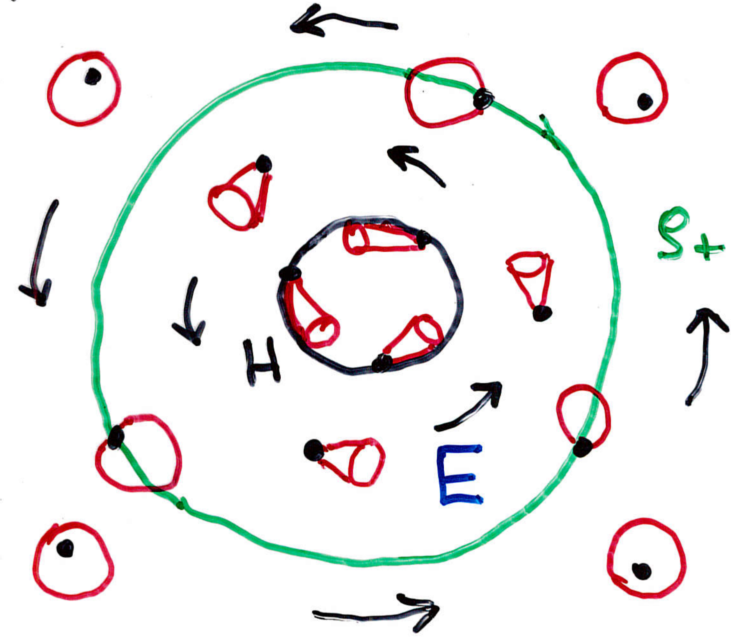
$$u^a = \frac{1}{|\nabla t|} \nabla^a t$$

the coord. angular velocity:

$$\Omega := \frac{d\phi}{dt} = \frac{a(r^2+a^2-\Delta)}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta} \rightarrow \Omega_H$$

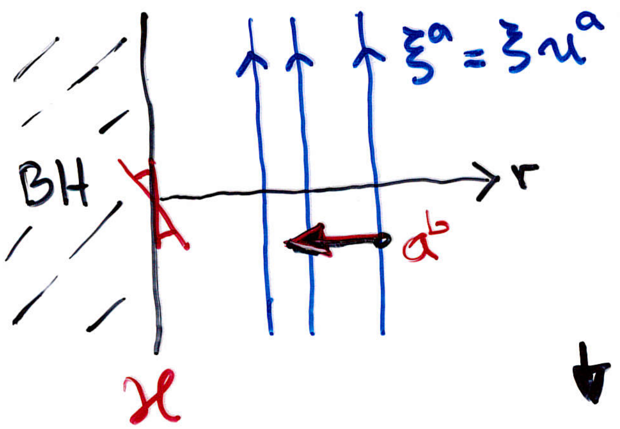
as $r \rightarrow r_+$ – NO observer being at rest in E

"top view in 3d"



• The surface gravity:

Event horiz. $\mathcal{H}$ – generated by null geodesics with tangent ξ^a to $\mathcal{H}$



$$\Rightarrow \quad \xi^a \nabla_a \xi^b \big|_{\mathcal{H}} =: \kappa\, \xi^b$$

"surface gravity"

$$\xi^a \nabla_a \xi^b = \xi^2 u^a \nabla_a u^b = \xi^2 a^b$$

$$\Rightarrow \quad \kappa = \lim(\xi |a^b|) = \frac{\sqrt{m^2-a^2-e^2}}{2m\left(m+\sqrt{m^2-a^2-e^2}\right)-e^2}$$

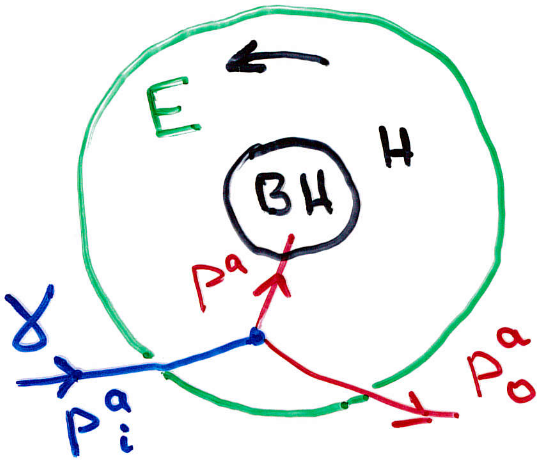
zero for extreme KN

- The electric potential w.r.t. ξ^a:

$$\bar{A}_a \xi^a = -\frac{er}{R^2}\,\frac{r_+^2 + a^2\cos^2\theta}{r_+^2 + a^2} \longrightarrow \mathcal{U} := -e\,\frac{r_+}{r_+^2 + a^2}$$

- The PENROSE process: $p^a = \mu u^a$ – test particle

$$\left.\begin{aligned} E &:= K_a p^a = \text{const.} \\ L &:= X_a p^a = \text{const.} \end{aligned}\right\} \text{along the geod. world line}$$



$\exists$ initial conditions for γ such that

$$E_i - E_o = K_a(p_i^a - p_o^a) = K_a p^a < 0$$

– energy extraction from KN black hole!

- The Kerr-Newman black hole state space:

$$\{(m, a, e) \,,\, m^2 \geq a^2 + e^2\} \text{ - solid cone in } \mathbb{R}^3$$

The area of $H := \mathcal{H} \cap \Sigma$

$$A = \mathrm{Area}(H) = 4\pi\left(2m^2 + 2m\sqrt{m^2 - a^2 - e^2} - e^2\right)$$

Its inverse for m in terms of A, $J := ma$, e:

$$m^2 = \frac{\pi}{A}\left(\left(\frac{A}{4\pi} + e^2\right)^2 + 4J^2\right)$$

$$\downarrow$$

$$dm = \frac{\kappa}{2\pi}\frac{dA}{4} + \Omega_H \, dJ - \mathcal{U} \, de$$

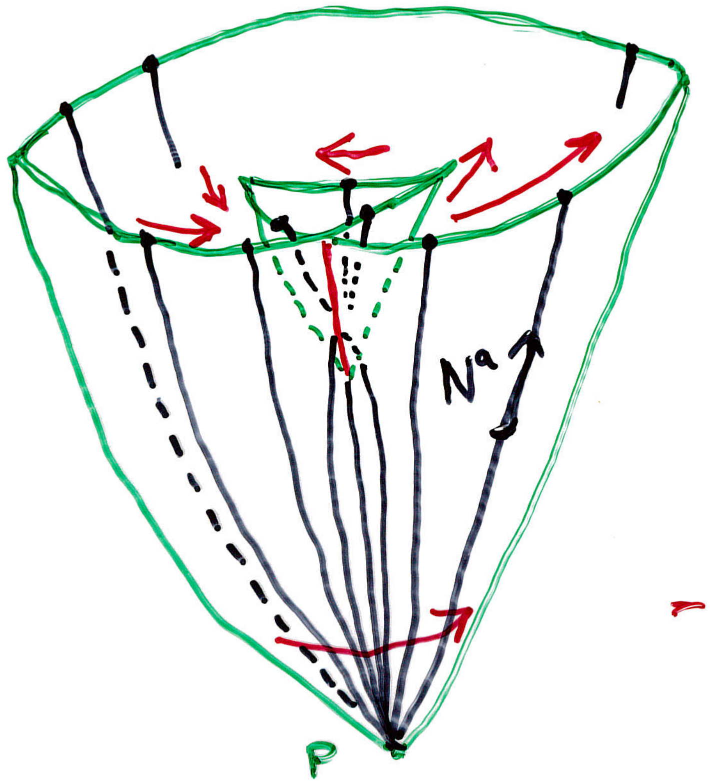
$$\left.\begin{array}{l} \text{- "stationary states version"} \\ \text{- "phys. process version"} \end{array}\right\} \text{ of the 1st law}$$

NB.: In the Penrose process $dA = 0$.

- Significance of KN:
  - uniqueness: no more asympt. flat, stationary, axi-symm. solution of the Einstein-Maxwell eqs. with regular event horizon
  - final state conjecture: final state of a generic grav. collapse is Kerr-Newman

Most of the basic notions of classical BH phys. can be illustrated by KN.

3. Geodesic focussing



- null geodesic generators of $\partial I^+(p)$ from p

θ - expansion, σ_{mn} - shear

$$\dot{\sigma}_{mn} = -\theta\sigma_{mn} + C_{manb}N^aN^b$$

$$\dot{\theta} = -\frac{1}{2}\theta^2 - \sigma_{mn}\sigma^{mn} + R_{ab}N^aN^b$$

If $R_{ab}N^aN^b \leq 0$, $\exists u_0 > 0 : \theta(u_0) < 0$, then
$\exists\, u_0 < u < u_0 - 2/\theta(u_0) : \theta(u) = -\infty$

- null geod. focussing in finite affine length
- caustic surfaces appear

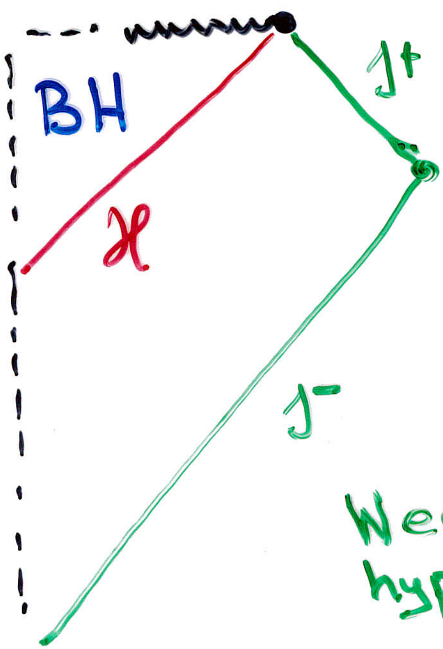
4. Black holes and the area law ("2nd Law")

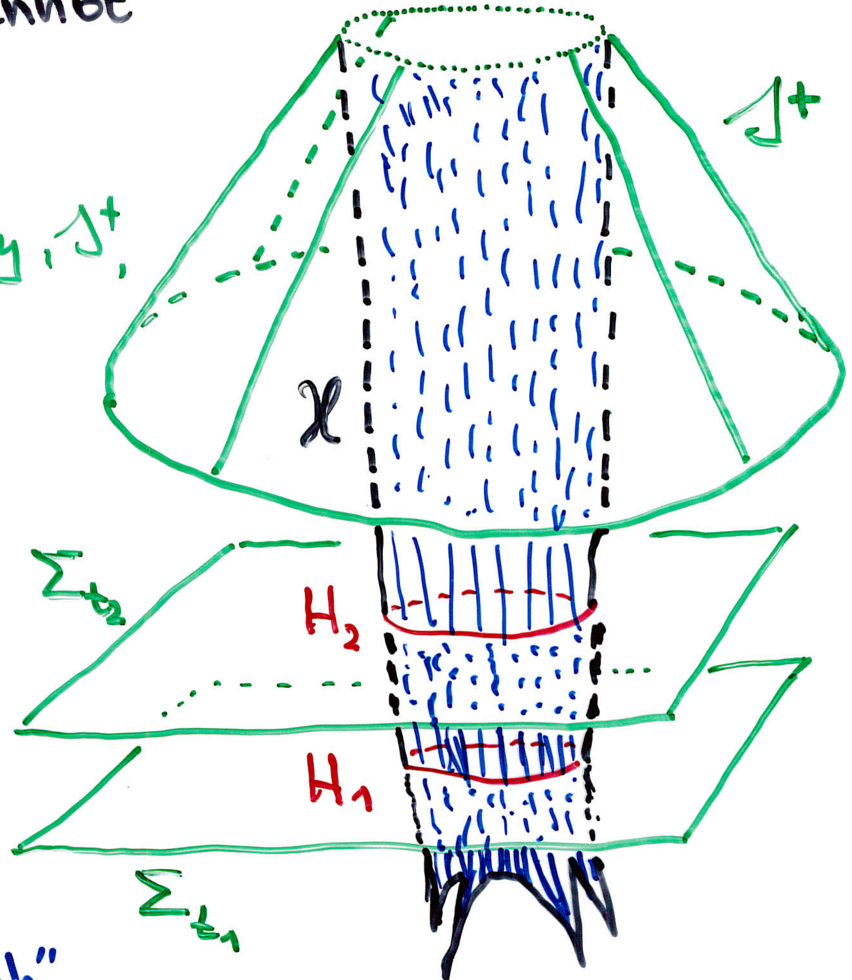
Idea: BH – domain whose points cannot be seen from infinity

Math. formulation: in spacetimes admitting future null infinity, $\mathcal{J}^+$,

$$BH := M - I^-[\mathcal{J}^+, \bar{M}]$$

$$\mathcal{H} := \partial I^-[\mathcal{J}^+, \bar{M}]$$





Lemma: $\mathcal{H}$ is generated by null geodesics with no future endpoint.

Weak cosmic censorship hypothesis: The outside regime is "regular enough" ($I^-[\mathcal{J}^+, \bar{M}] \cup$ (neighb. of $\mathcal{H}$) is globally hyperbolic)

Theorem (Hawking): If WCCH holds and $T_{ab}N^a N^b \geq 0$, then $\theta \geq 0$ holds for the null geod. generators of $\mathcal{H}$.

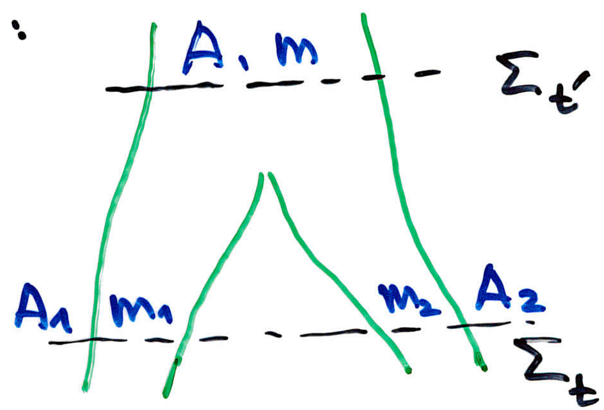
Horizon in the 3-space Σ: $H := \mathcal{H} \cap \Sigma$

Then: $\text{Area}(H_{t_1}) \leq \text{Area}(H_{t_2})$, $t_1 \leq t_2$ (2nd Law).

- Energy extraction from BH coalescence:

By the area thm: $A \geq A_1 + A_2$

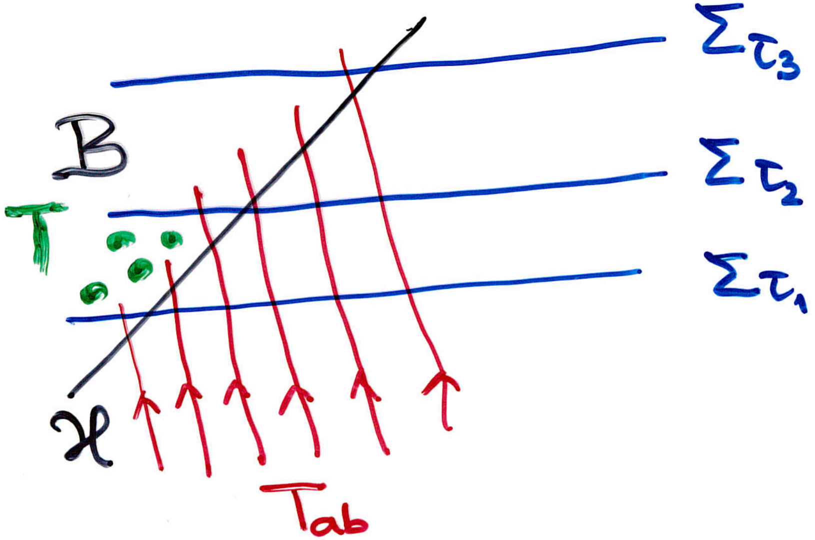
Area thm + final state conjecture
$\rightarrow$ upper bound for the energy radiated away:



$$\varepsilon := \frac{m_1 + m_2 - m}{m_1 + m_2} \leq \begin{cases} 1 - \frac{1}{\sqrt{2}} & \text{for Schwarzschild} \\ \frac{1}{2} & \text{for two extreme Kerr} \\ 1 - \frac{1}{2\sqrt{2}} & \text{extreme KN, RN} \end{cases} .$$

5. The '3rd Law'

- 'It is impossible, by any process, to reduce κ to zero in a finite sequence of operations' – ill defined concepts
- Reformulation: We saw:



non extreme KN black hole ($\kappa \neq 0$) $\Leftrightarrow$
$\Leftrightarrow$ $\exists$ T closed trapped surfaces just behind the horizon $\mathcal{H}$

Theorem (Israel): If $\exists$ Σ_τ Cauchy surfaces with closed trapped surfaces in $B \cap \Sigma_\tau$ for $\tau < \tau_2$
and with no trapped surf. in $B \cap \Sigma_\tau$ for $\tau > \tau_2$,
then $T_{ab} N^a N^b \geq 0$ cannot hold for any N^a.

- "non-extreme b.h. cannot become extreme in finite advanced time in any continuous process with bounded normal matter"
- dynamical, "phys. process version", like the "2nd Law".

6. Stationary black holes (the '0th and 1st Laws')

- Structure of stationary black hole spacetimes:

Theorem (Hawking):
Let K^a be a Killing vector of stationarity
$\mathcal{H}$ - event horizon of the black hole.
Then either:
- K^a is static, no ergosphere, it is null tangent to $\mathcal{H}$

or
- there is an ergosphere, $\exists X^a$ axi-symm. Killing, $[K,X]^a = 0$, $\exists \Omega_H$: $\xi^a := K^a + \Omega_H X^a$ timelike Killing and null tangent to $\mathcal{H}$.

— analogous to Kerr-Newman!

- Definition of surface gravity κ:

$$\xi^a \nabla_a \xi^b \big|_{\mathcal{H}} = \kappa \xi^b$$

— like in the KN case

(Bardeen - Carter - Hawking):

Theorem: If Einstein's eqs. hold + T_{ab} satisfies the dominant energy cond. (i.e. $T^a{}_b V^b$ is future causal for any future causal V^a), then

$$\kappa = \text{const} \quad \text{on } \mathcal{H}$$

also: $E^a := F^a{}_b \, \xi^b \big|_{\mathcal{H}} \sim \xi^a$, $\rightarrow$ the electric potential

$$\mathcal{U} = \text{const} \quad \text{on } \mathcal{H}$$

— 'Zeroth Law'

- The '1st Law':

Total mass at infinity: $M := \frac{1}{8\pi G} \oint_{S_\infty} \nabla^{[a} K^{b]} \, dS_{ab} =$

$$= \int_\Sigma 2\left(T_{ab} - \tfrac{1}{2} T g_{ab}\right) K^a t^b \, d\Sigma + \frac{1}{8\pi G} \oint_H \nabla^{[a} K^{b]} \, dS_{ab} =$$

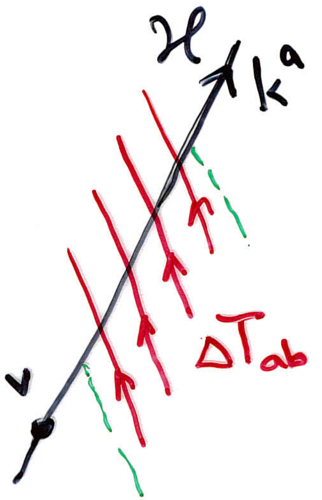
$$= \int_\Sigma 2\left(T_{ab} - \tfrac{1}{2} T g_{ab}\right) K^a t^b \, d\Sigma + 2\,\Omega_H J_H + \frac{\kappa}{2\pi} \frac{A}{4G}$$

— geom. identity

$$\delta M = \frac{\kappa}{2\pi}\,\delta\left(\frac{A}{4G}\right) + \Omega_H\,\delta J_H + \mathcal{U}\,\delta Q$$

- formal, "stationary states version" of the '1st Law'

• The 'physical process version' of the 1st Law:



$$\left.\begin{aligned} \Delta M &:= \int_0^\infty \oint_{S_v} \Delta T_{ab} K^a k^b \, d\$ \, dv && \text{– energy} \\ \Delta J &:= \int_0^\infty \oint_{S_v} \Delta T_{ab} X^a k^b \, d\$ \, dv && \text{– ang. mom.} \end{aligned}\right\} \text{flux into the BH}$$

The resulting area increase:

$$\Delta A = \int_0^\infty \frac{d}{dv} \oint_{S_v} d\$ \, dv = \int_0^\infty \oint_{S_v} \theta \, d\$ \, dv$$

– the same equality for these, too.

Consistency of the two (logically different) derivations!

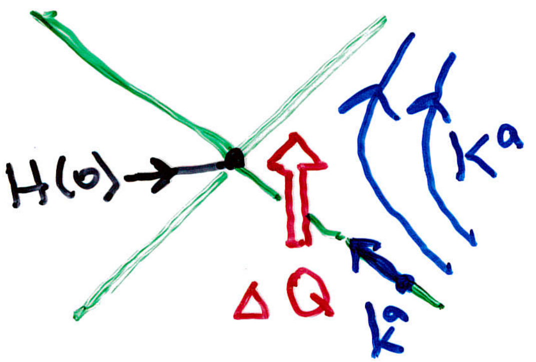
Open questions: interpretation of M, J!

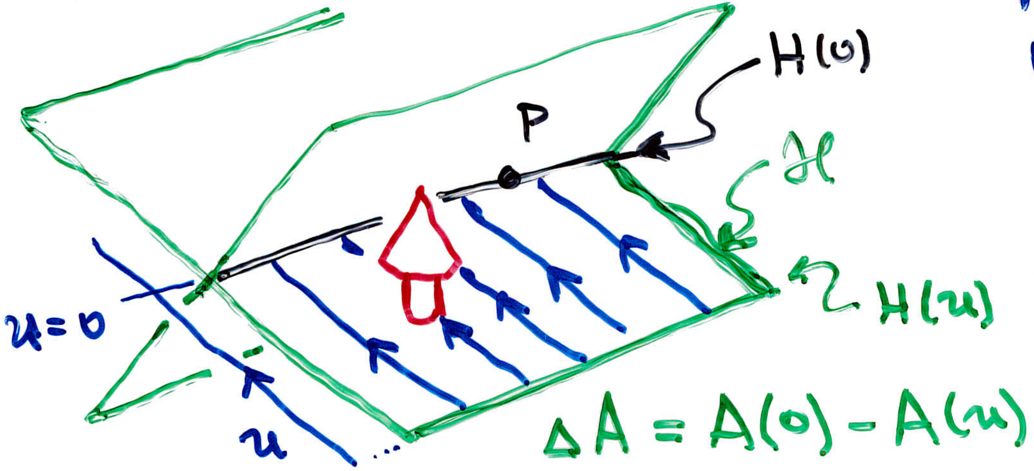
7. GR from the 1st Law

- Jacobson: associate $S = \frac{1}{4G}\,\text{Area}(H)$ entropy, and $T = \frac{\kappa}{2\pi}$ temperature to any local Rindler horizon, and postulate the 1st Law, $\Delta Q = T \Delta S$, with ΔQ the energy flux through H seen by a Rindler observer

$\Downarrow$ Einstein's equations follow.

- Local Rindler horizon (at equilibrium):





Approx. boost Killing field:

$K^a = -u\kappa k^a$

where $k^a \nabla_a k^b = 0$

'Equilibrium': $\theta = 0,\ \sigma_{mn} = 0$ on $H(0)$

- The derivation of Einstein's eqs.:

Roychaudhuri eq.: $\dfrac{d\theta}{du} = -\frac{1}{2}\theta^2 - \sigma_{mn}\sigma^{mn} + R_{ab}k^a k^a \simeq R_{ab}k^a k^b$

$\downarrow$

$\theta = u R_{ab} k^a k^b$

$\downarrow$

$\Delta A = \int_u^0 \dfrac{dA}{du}\, du = \int_{\mathcal{H}} \theta \, dA\, du = \int_{\mathcal{H}} u R_{ab} k^a k^b \, dA\, du$

$\downarrow$

$T\Delta S = \dfrac{\kappa}{8\pi G} \Delta A = \dfrac{1}{8\pi G} \kappa \int_{\mathcal{H}} u R_{ab} k^a k^b \, dA\, du$

and

$\Delta Q := \int_{\mathcal{H}} T_{ab} K^a \, d\mathcal{H}^b = -\int_{\mathcal{H}} T_{ab} u \kappa k^a k^b \, dA\, du$

$\downarrow$

$R_{ab} k^a k^b = -8\pi G T_{ab} k^a k^b$ for any null k^a

$\downarrow$

$R_{ab} + f g_{ab} = -8\pi G T_{ab}$

By the twice contracted Bianchi identity + $\nabla_a T^a{}_b = 0$

$\hookrightarrow$ $R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = -8\pi G T_{ab}$.

8. The generalized 2nd Law (GSL)

- Throwing matter into BH $\longrightarrow$ degrees of freedom, information are lost $\Rightarrow$ violation of the 2nd Law of thermodyn.
- Bekenstein: Associate entropy S_{BH} to BH and keep
$$\Delta(S_{mat} + S_{BH}) \geq 0.$$

Question: How to define S_{BH}?

Hawking's area theorem $\longrightarrow$
$$S_{BH} = k_B\, f\left(\frac{\text{Area}(H)}{4 L_p^2}\right), \quad L_p^2 := \frac{G\hbar}{c^3}$$

f: strictly mon. increasing function, but. e.g. $f(x) \sim \sqrt{x}$ is NOT good – restriction from spec. BH cases

Plausible candidate: $S_{BH} = \eta\, k_B \dfrac{\text{Area}(H)}{4 L_p^2}$ $\quad \left(\eta = \dfrac{\ln 2}{2\pi^2}\right)$

NB: $S_{\odot} \sim 10^{42}\, \text{erg}/K^{\circ}$, $S_{BH\odot} \sim 10^{60}\, \text{erg}/K^{\circ}$

Claim (GSL): $\Delta(S_{mat} + S_{BH}) \geq 0$ in any process

9. The Hawking temperature

- In traditional units the 1st Law:

$$dE = \frac{c^2}{G}\frac{\kappa}{2\pi}\, d\left(\frac{A}{4}\right) + \dots = T_{BH}\, dS_{BH} + \dots$$

with $S_{BH} = \eta\, k_B \dfrac{\text{Area}(H)}{4\,(G\hbar/c^3)}$, $T_{BH} = \dfrac{1}{\eta}\dfrac{\hbar}{k_B c}\dfrac{\kappa}{2\pi}$ $\left(\eta = \dfrac{\ln 2}{2\pi^2}\right.$ – Bekenstein's choice$\left.\right)$

Bardeen – Carter – Hawking:

'If S_{BH} were physical entropy, then T_{BH} would have to be phys. temperature, and BHs would have to be radiating – non sense!

- Hawking radiation: thermal Planck spectrum with temperature

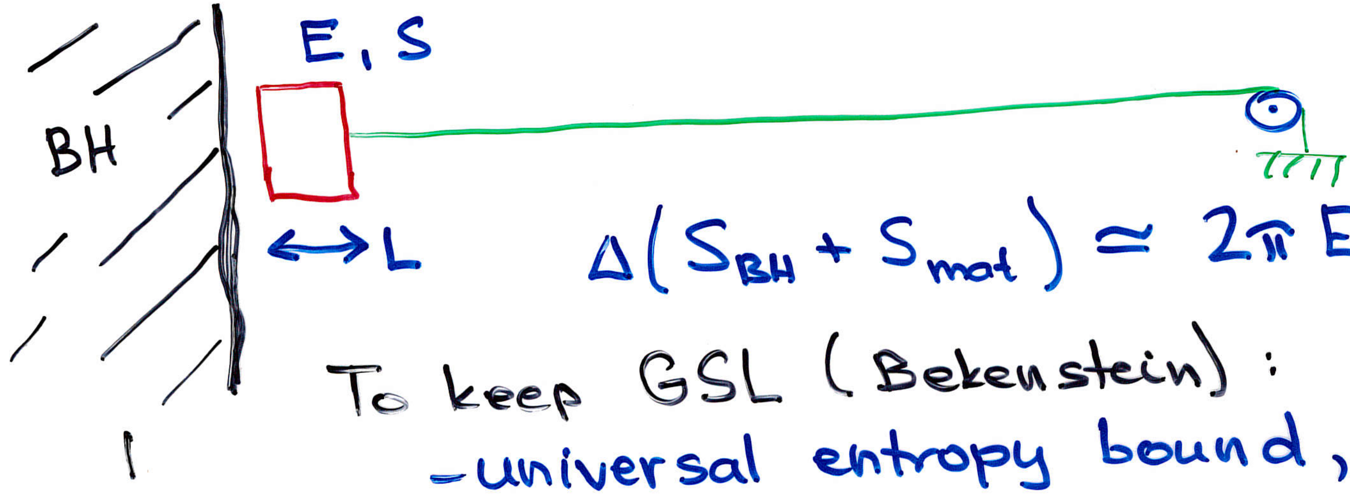
$$T_H = \frac{\hbar}{k_B c}\frac{\kappa}{2\pi} \sim 6 \times \frac{M_\odot}{M} \times 10^{-8}\ K^\circ$$

$\downarrow$

- ~~Both~~ T_{BH}, S_{BH} are physical!
- BHs have negative heat capacity $\longrightarrow$ instability: BH evaporation is accelerating, ends with a 'pop' (like the explosion of an artillery ball)
- $\eta = 1$.

10. The universal entropy bound

- The Geroch process:



$$\Delta(S_{BH} + S_{mat}) \simeq 2\pi E L - S$$

To keep GSL (Bekenstein): $S \leq 2\pi E L$

– universal entropy bound, with NO G!

- Debates: – how to define L – ambiguity (Page)
  – no violation of GSL by Hawking rad. (Unruh-Wald)

- Consequences of the bound:
  1. bound on shear viscosity: $\eta/s \geq \frac{1}{4\pi}$ (Hod)
  2. bound on relaxation time: $T\tau \geq \frac{1}{\pi}$ (Pesci)

  ⋮

  n. bound on electric/heat conductivity ?

  – no ideal materials ? ?