Anderson transition in the spectrum of the Dirac operator

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Collaboration

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Continuum QCD

- Field theoretic description of the strong interaction
- A gluon field, ψ quark field
- Lagrangian:

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left(\not{D} + m \right) \psi$$

- QCD observable:

$$\langle O \rangle = \int \mathrm{d}A \mathrm{d}\bar{\psi} \mathrm{d}\psi O \exp\left(i \int L \, \mathrm{d}^4 x \right)$$

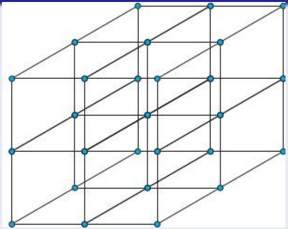
- On Hadronic energy scales QCD is strongly interacting
- Non perturbative regulator: lattice

Lattice QCD

- 4d hypercubic lattice
- Quark fields on lattice sites
- Gluons on links
- Euclidean time ($t_E = -it$)

Lattice discretization

- $\psi(x_i)$ 3d complex vector
- $U_\mu(x_i)$ SU(3) group element



Lattice QCD

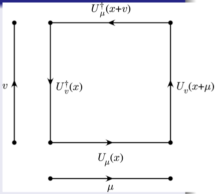
- Observable in LQCD:

$$\langle O\rangle = \frac{1}{Z}\int \mathrm{d}\psi(x)\,\mathrm{d}\bar{\psi}(x)\,\mathrm{d}U_\mu(x)\,O(\psi,\bar{\psi},U)\exp\left(-S(\psi,\bar{\psi},U)\right)$$

Interactions

- Among glouns:

$$U_\mu(x)\,U_\nu(x+\mu)\,U^\dagger_\mu(x+\nu)\,U^\dagger_\nu(x)$$



- Between quarks:

$$D(x,y) = \sum_\mu \eta_{\mu,x}\left(U_\mu(x)\,\delta_{y,x+\mu} - U^\dagger_\mu(x-\nu)\delta_{y,x-\mu}\right)$$

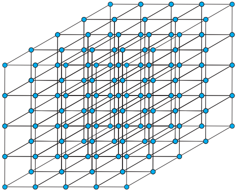
- Question: Why can I write “almost everything” in the Lagrangian?

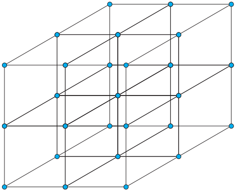
Taking the continuum limit: Wilson RG

- Let's have a very complicated lattice action: $S = \sum_i c_i S_i(U)$
- Let's do a blocking transformation $U_\mu(x) \to W_\mu(2x)(U)$

$$Z = \int \mathrm{d}U \exp(-S(U)) = \int \mathrm{d}U \mathrm{d}V \prod_x \delta(V(U_x) - W_\mu(U_x)) \exp(-S(U))$$

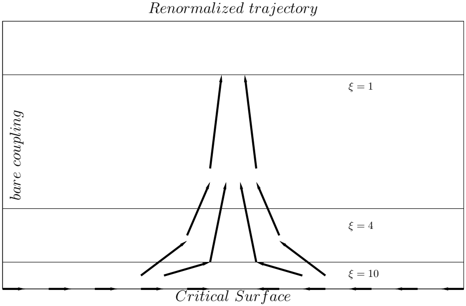
$$= \int \mathrm{d}V \exp(-S_{new}(V))$$





- Long distance physics unchanged
- $a_{new} = 2a$
- $\xi_{new} = \frac{\xi}{2}$

Taking the continuum limit
Fixed points and renormalized trajectory



- Simple example: just two couplings
- $SU(3)$ pure gauge: β
- Irrelevant flows to the fix point
- Relevant has to be tuned
- Continuum limit: decreasing bare coupling, maintaining the long distance physics
- Fermions present: Give up one physical quantity for every relevant parameter everything else is a prediction

Lattice Dirac operator

- Quarks can be integrated out explicitely in the partition function:

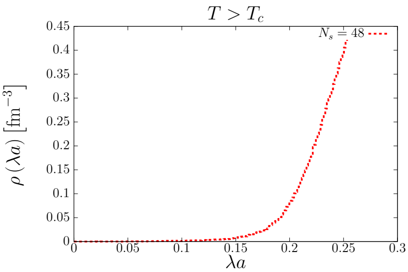
$$Z = \prod_{x,\mu} \mathrm{d}U_\mu(x) \det(D(U)+m) \exp\left(-S_g(U)\right)$$

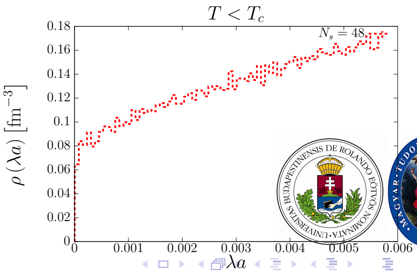
D(U): Massless lattice Dirac operator

- Discretization of the continuum $\gamma_\mu\left(\partial_\mu + A_\mu\right)$
- Observables can be built up from eigenmodes of D
- D can be treated formally as a random “Hamiltonian”: $P(U) \propto \exp(-S_g(U))\det(D(U)+m)$

QCD Chiral transition

- Order parameter of chiral symmetry: Banks-Casher (T. Banks and A. Casher (1980))
  $\langle \bar{\psi}\psi \rangle = \lim_{m \to 0} \lim_{V \to \infty} \frac{\pi \rho(0)}{V}, \rho(\lambda) = \langle \sum_i \delta(\lambda - \lambda_i) \rangle$
- Low eigenvalues connected the chiral symmetry breaking
- $\rho(\lambda = 0) \neq 0$ means non zero condensate





Dirac spectrum

Below T_c: Random Matrix Theory (*RMT*)

- Eigenmodes follow Random Matrix statistics
- Random Matrix model can be constructed where the elements of D are independently identically distributed random variables.
- Analytic and numerical connection to QCD in the so-called ε-regime Shuryak, Verbaarschot (1992): $\frac{1}{\Lambda_{QCD}} \ll L \ll \frac{1}{m_\pi}$

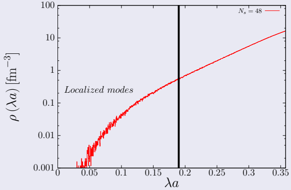
Above T_c

- The spectral statistics will depend on the position in the spectrum
- Above T_c the low modes become localized, still correlations in the bulk show *RMT* behaviour García-García and Osborn(2006);T. G. Kovács(2010);R. Pullirsch, K. Rabitsch, T. Wettig and H. Markum(1998)

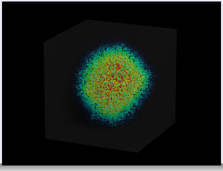
Low end of the spectrum

Mixing is small

- Number of modes is small

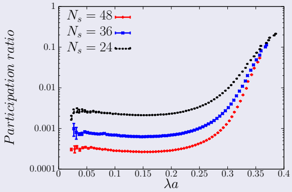


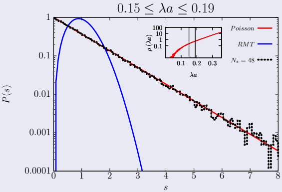
- Modes are spatially localized



Poisson statistics, localized modes

- Unfolded level spacing distribution: $s = \frac{\lambda_{n+1} - \lambda_n}{\langle \lambda_{n+1} - \lambda_n \rangle}$

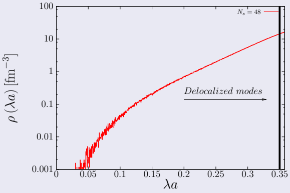




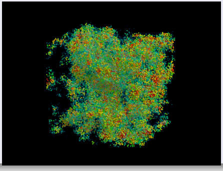
Bulk of the spectrum

Mixing is large

- Number of modes is large

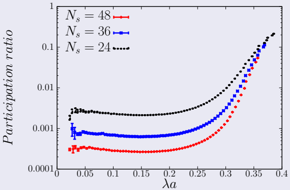


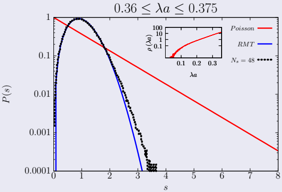
- Modes are spatially delocalized



Random-Matrix statistics, delocalized modes

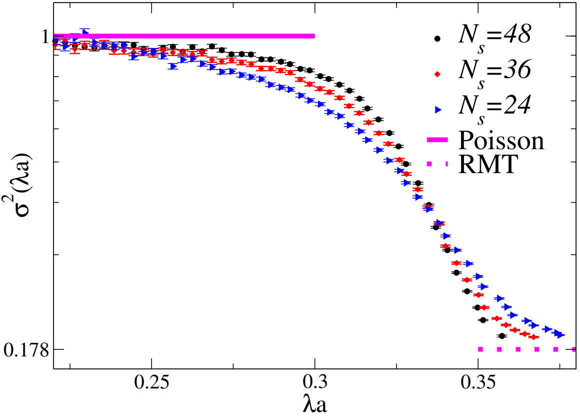
- Unfolded level spacing distribution: $s = \frac{\lambda_{n+1} - \lambda_n}{\langle \lambda_{n+1} - \lambda_n \rangle}$





What happens in between these two parts?

Variance of the ULSD in the spectrum for $n_t = 4, a = 0.125\mathrm{fm}$



- We can define a mobility edge(λ_c) for all V
- $\lambda_c \equiv$ Location of the inflection point of the curves

? What happens in the continuum limit?

? Other models showing the same thing?

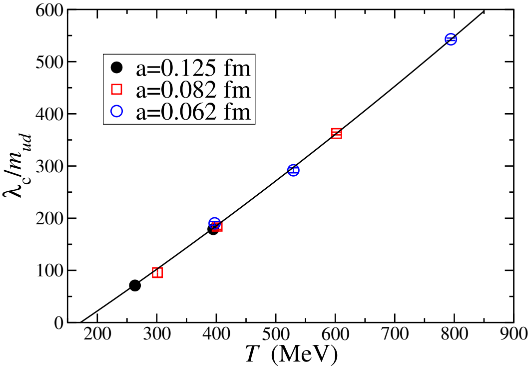
? Is it a real phase transition?

Mobility edge in the continuum limit

T.G. Kovács and FP Phys. Rev. D (2012)

How λ_c behaves in the continuum?

- λ_c introduces an effective gap in the spectrum
- Its renormalization is similar to the quark mass
- λ_c / m_{ud} tends to finite value in the continuum limit
- Localization is physical



- $\lambda_c(T_c) = 0 \rightarrow T_c \simeq 170$ MeV

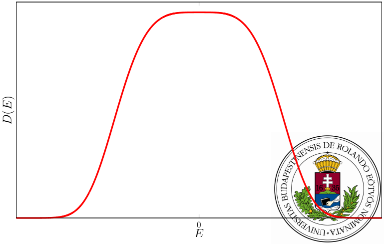
Anderson tight binding model

Non interacting electrons with one electron Hamiltonian

$$H = \sum_i \varepsilon_i |i\rangle\langle i| + \sum_{(i,j)} |i\rangle\langle j|$$

- $|i\rangle$ atomic orbital with energy ε_i
- Second sum represents the hopping of the electron to the neighboring site
- Eigenstates are delocalized Bloch waves

- Impurity: $\varepsilon_i \to$ random
- localized modes appear at the spectrum edge
- $\sigma(\varepsilon_i)$ grows $\to$ mobility edge moves towards the band



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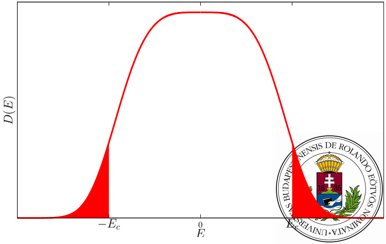

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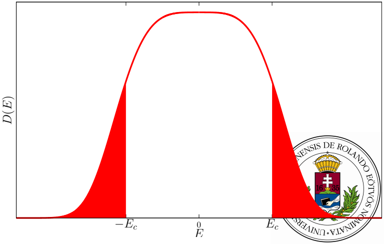

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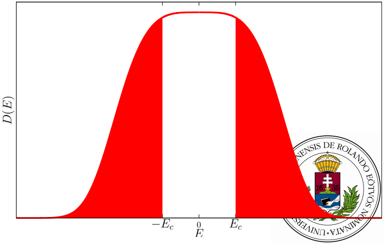

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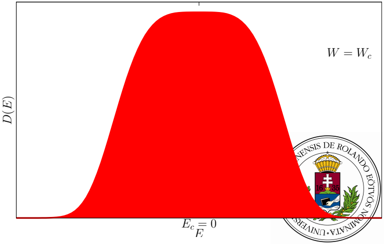
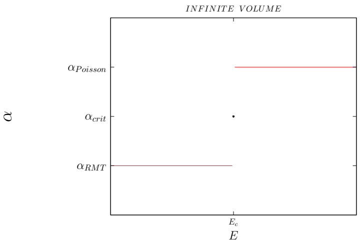


Anderson transition

Finite size scaling analysis B. Kramer, A. MacKinnon, T. Ohtsuki and K. Slevin (2010)

- In 3d it is a real second order phase transition
- Consider a dimensionless physical quantity $\alpha\,(E, s, \cdots)$
- In the thermodynamic limit:

$$\alpha\,(E, s, \cdots) = \begin{cases} \alpha_{RMT} & \text{for} \quad E \leq E_c \\ \alpha_{CRIT} & \text{for} \quad E = E_c \\ \alpha_{Poisson} & \text{for} \quad E \geq E_c \end{cases}$$



- Finite volume: One parameter scaling hypothesis
- α depends on:
  1. L: system size
  2. E: relevant variable
  3. s: leading irrelevant variable

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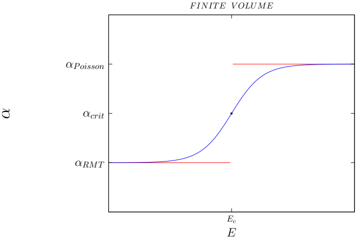

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Anderson transition

Finite size scaling analysis, Renormalization Group argument

- $\alpha\left(E,s,L^{-1}\right)$ is a dimensionless physical quantity, it should be invariant under renormalization group transformation with scale factor ℓ:

$$RG(l):\begin{cases} E-E_c & \rightarrow & \ell^{\frac{1}{\nu}}\cdot\left(E-E_c\right), \frac{1}{\nu}>0 \\ s-s_{fp} & \rightarrow & \ell^{y}\cdot\left(s-s_{fp}\right), y<0 \\ L^{-1} & \rightarrow & \ell\cdot L^{-1} \end{cases}$$

$$\alpha\left(E-E_c, s-s_{fp}, L^{-1}\right)=\alpha\left(\ell^{\frac{1}{\nu}}\left(E-E_c\right), \ell^{y}\left(s-s_{fp}\right), \ell\, L^{-1}\right)$$

- Fix the RG scale such that $\ell\cdot L^{-1}=c$

$$\alpha\left(E-E_c, s-s_{fp}, L^{-1}\right)=\alpha\left((cL)^{\frac{1}{\nu}}\left(E-E_c\right),(cL)^{y}\left(s-s_{fp}\right),1\right)$$

$$\alpha\left(E-E_c, s-s_{fp}, L^{-1}\right)=f_c\left(L^{\frac{1}{\nu}}\left(E-E_c\right), L^{y}\left(s-s_{fp}\right)\right)$$

Anderson transition in the Dirac spectrum

Extraction of the critical exponents

Transition in the Dirac spectrum

- Near the critical point a dimensionless local physical quantity Q does not depend separately on λ and L(lattice extension): $Q(\lambda, L) = f\left(\frac{L}{\xi_\infty(\lambda)}\right)$
- Using the definition of the correlation length $\left(\xi_\infty(\lambda) \propto (\lambda - \lambda_c)^{-\nu}\right)$ we can expand Q around λ_c for λ close to λ_c:

$$Q(\lambda, L) = f\left(\frac{L}{\xi_\infty(\lambda)}\right) = F\left(L^{\frac{1}{\nu}}(\lambda - \lambda_c)\right) =$$

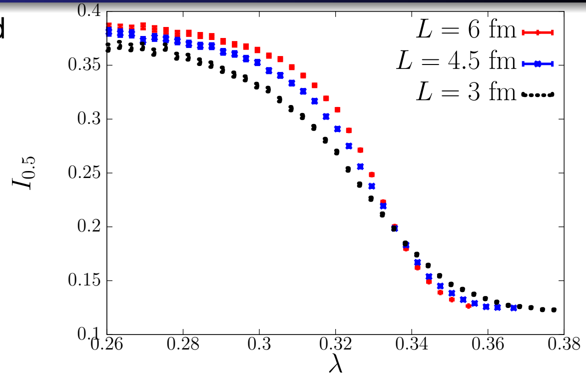
$$= \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} L^{\frac{n}{\nu}} (\lambda - \lambda_c)^n$$

- determining λ_c, ν using the coefficients of a polinomial fit

Anderson transition in the Dirac spectrum

Critical exponents and Data collapse

- Measuring λ_c and ν using several volumes in two variables fit
- Estimate the systematic error through constrained fits including more and more terms in the expansion
- We use the quantity $I_s(\lambda) = \int_0^s p(s')\, ds'$



- Critical exponents consistent with the 3d unitary Anderson model:

$$\nu = 1.435(59), \lambda_c = 0.33604(37)$$

M. Giordano, T.G. Kovács and FP (PRL 2014)

Anderson transition in the Dirac spectrum

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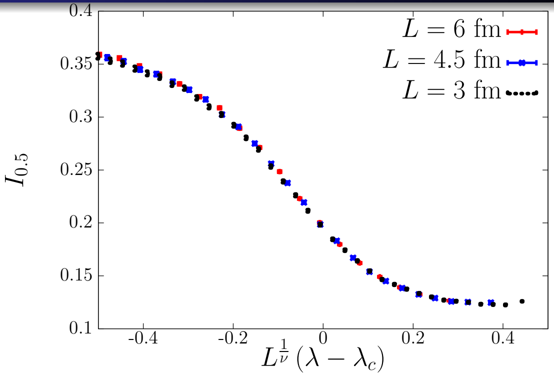

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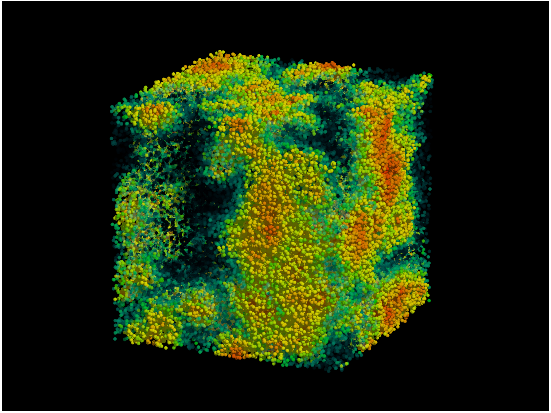
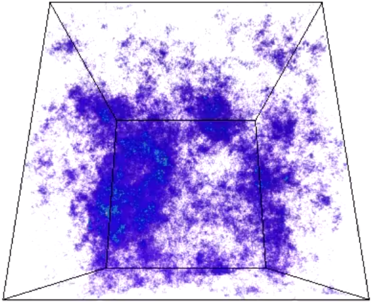
How the eigenvectors look like around the mobility edge?



Inside a multifractal at the Anderson transition

Schenk, Bollhöfer, Römer, SIAM Reviews 50, 91-112 (2008)]



Multifractals: Characterized by infinite number of fractal dimensions

- Let $|\psi(x)|^2$ give the probability of finding a particle in an infinitesimal region around x.
- Finding the particle within a d dimensional sphere of radius r (r small) scales as:
$$r^d$$
- For fractal wavefunctions this scales as
$$r^\alpha$$
where $\alpha < d$ is the fractal dimension.
- For strongly fluctuating wavefunctions we can define local fractal dimensions
$$r^{\alpha(x)}$$

Multifractal exponents

- Divide our 4d lattice into smaller boxes of linear size ℓ.
- Probability for boxes: $\mu_k = \sum_{x \in \mathrm{box}_k} |\psi(\vec{x})|^2$
- From the moments of box probabilities one can define generalized inverse participation ratios:

$$R_q = \sum_{k=1}^{\Lambda^{-d}} \mu_k^q, \; \Lambda = \frac{\ell}{L}$$

- Obtain multifractal exponents check how R_q scales with Λ (small):

$$R_q \propto \Lambda^{-D_q(q-1)}$$

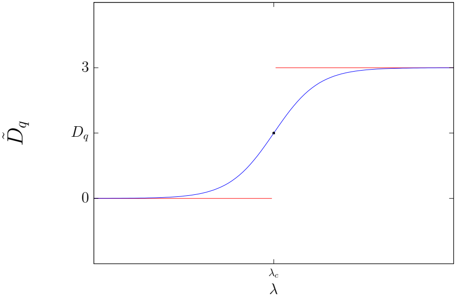
Multifractal exponents

- For $q = 2$ we obtain the "effective" volume occupied by the eigenmode
- For a localized mode $\propto \Lambda^0 \rightarrow D_{q=2} = 0$
- For a delocalized mode $\propto \Lambda^3 \rightarrow D_{q=2} = 3$
- D_q does not depend on q for (de)localized mode
- At λ_c modes are expected to be a multifractal:
  D_q will depend on q
- One can define the ensemble average of D_q in a specific spectral window:

$$\tilde{D}_q(\lambda, L, \ell) = \frac{1}{q-1} \frac{\ln\langle R_q \rangle}{\ln\Lambda}$$

Finite size scaling for multifractal exponents

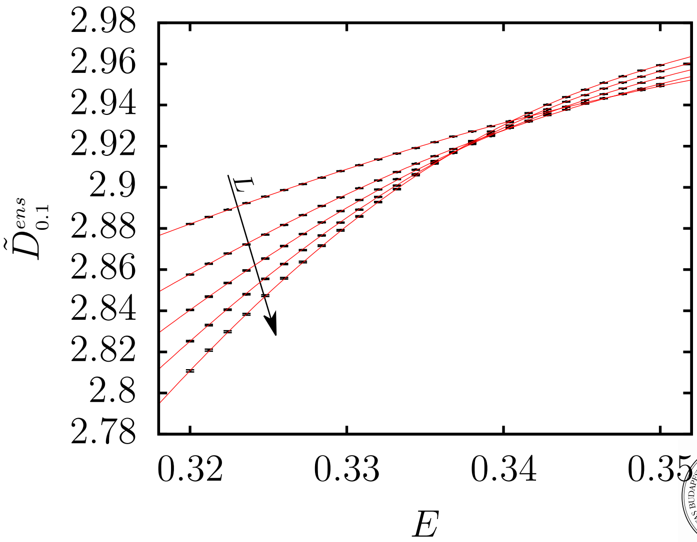
- In the spectrum around λ_c D_q will jump in the thermodynamical limit



- Finite size scaling formula

$$\tilde{D}_q(\lambda, L, \ell) = D_q + \frac{1}{\ln \Lambda} f\left(\frac{L}{\xi}, \frac{\ell}{\xi}\right)$$

Finite size scaling at fixed $\Lambda = \frac{1}{8}$



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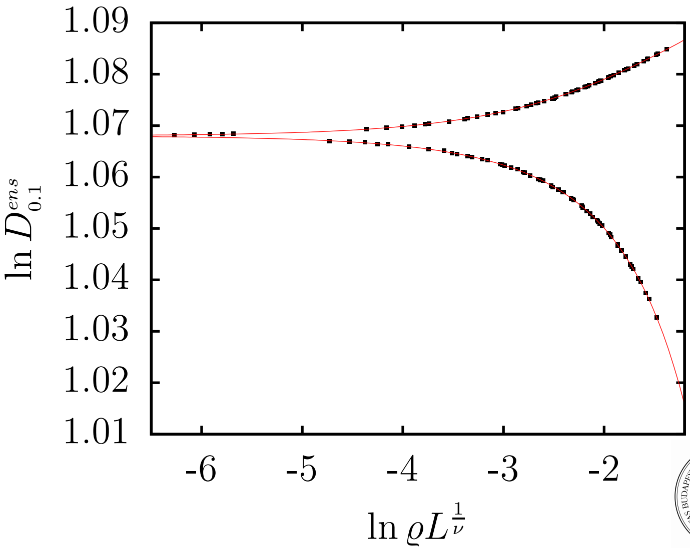
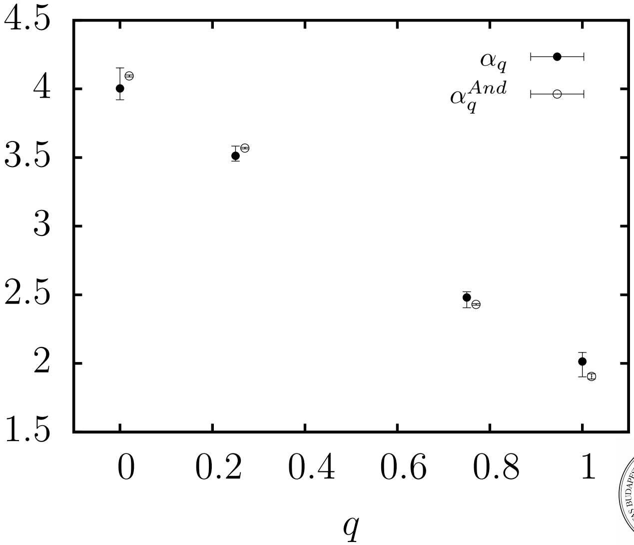


Multifractal exponents for the QCD Dirac spectrum



Conclusions

- There are two different models: QCD and Anderson tight binding
- Differences
  - Scales of the problem
  - Non-zero elements
  - Correlations among the elements
- Yet they belong to the same universality class, share the same multifractal exponents
- For more details see:
  L. Ujfalusi, M. Giordano, FP, T. G. Kovács and I. Varga,Phys. Rev. D **92**, no. 9, 094513 (2015)
- Thank you for attention!