The Julia Set of Perturbed Quadratic Maps $F_\lambda(z) = z^2 + c + \lambda/z^2$ Converging to the Filled Quadratic Julia Sets

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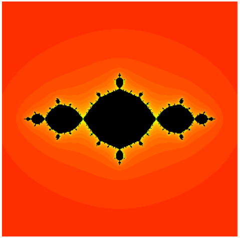
Outline

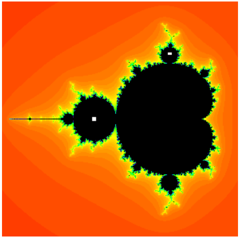
1. **Preliminaries**
   Julia set, Mandelbrot set, critical point, trap door
2. **Previous Results**
   Behavior of Julia sets under specific conditions
3. **Part I: A Basic Case**
   Julia sets converge to the filled basilica for $c = -1$
4. **Part II: General Case**
   Theorem: Julia sets converge to the filled Julia set when c lies at the center of a hyperbolic component of the Mandelbrot set
5. **Concluding Remarks**

Introduction

Even the simplest nonlinear functions exhibit intricate dynamical behavior:

$$F(z) = z^2 + c$$





- Understand what these objects are.
- Discuss some of their properties.

Introduction

Discrete dynamical system

$$F(z) = z^2 + c$$

Fix parameter c and iterate:

1. $F(z) = z^2 + c$
2. $F^2(z) = F(F(z)) = (z^2 + c)^2 + c$
3. $F^3(z) = F(F(F(z))) = ((z^2 + c)^2 + c)^2 + c$
4. and so on ...
5. $F^n(z) = \underbrace{F(F(\ldots F(z)..)}_{n}$

The sequence of numbers generated is called the **orbit of z**.

Introduction

What happens to the orbit as $n \to \infty$?

$$\mathcal{O} = \{z, F(z), F^2(z), F^3(z), \dots\}$$

- Periodic
- Eventually periodic
- Attracted to periodic n-cycle
- Diverge to infinity
- CHAOTIC

Introduction

What is CHAOS?

Definition (Devaney)

A dynamical system $(X, dist, F)$ is **chaotic** if:

(1) Periodic points are dense

(2) $F : X \to X$ is topologically transitive

(3) Sensitive dependence on initial conditions

Remark: (1) and (2) $\Rightarrow$ (3)

Introduction

What is CHAOS?

(1) Definition (Dense set)

A subset S of set X is **dense** if $\overline{S} = X$.

(2) Definition (Topologically Transitive)

$F : X \to X$ is topologically transitive if for any pair of open sets $U, V \subset X$ there exists $k > 0$ such that $F^k(U) \cap V \neq \varnothing$.

(3) Definition (Sensitive Dependence)

$F : X \to X$ has sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any $x \in X$ and any neighborhood N of x, there exists $y \in N$ and $n \geq 0$ such that $|F^n(x) - F^n(y)| > \delta$.

Introduction

Key Example:
Define the sequence space $\sum := \{s_1 s_2 s_3 s_4 \ldots | s_j = 0 \text{ or } 1\}$ with metric $d(\mathbf{s}, \mathbf{t}) := \sum_{i=1}^{\infty} \frac{|s_i - t_i|}{2^i}$. Define the **shift map** to be $\sigma : \sum \to \sum$ such that

$$\sigma(s_1 s_2 s_3 s_4 \ldots) \mapsto \sigma(s_2 s_3 s_4 \ldots).$$

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Claim: The shift map $\sigma : \sum \to \sum$ is a chaotic system.

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Proof.

Take binary representation of numbers in $[0, 1]$ after the decimal:

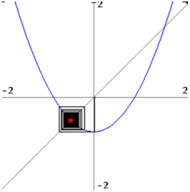
(1) Rationals are dense.

(2) Concatenate all rationals with finite binary representation.

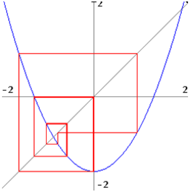
□

Iterates and Orbits of Quadratic maps $z^2 + c$

Graphical Analysis



$-\frac{3}{4} < c < \frac{1}{4}$



$c \approx -1.5$

Critical point

$$\frac{d}{dz}F(z) = 0$$

Critical orbit of $z^2 + c$

1. $F(0) = 0^2 + c$
2. $F(F(0)) = F(c) = c^2 + c$
3. $F(F(F(0))) = F(c^2 + c) = (c^2 + c)^2 + c$
4. Do it again ...

Background: 1910-1920's

Gaston Julia
1893-1978

Pierre Fatou
1878-1929

Preliminaries

Definition

The ***filled Julia set*** *is the set consisting of all points whose orbits stay bounded. The* ***Julia set*** *is the boundary of the filled Julia set and is denoted* $J(F_\lambda)$.

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The Julia set is the chaotic domain

Definition

The **Fatou set** is the complement of the Julia set.

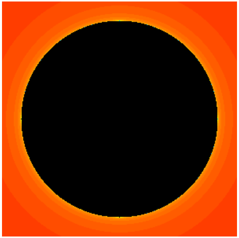
Preliminaries

Complex quadratic function $\mathbf{F(z) = z^2 + c}$

Iterates $F^2(z) = F(F(z)), \ldots, F^n(z) = F(F^{n-1}(z)), \ldots$

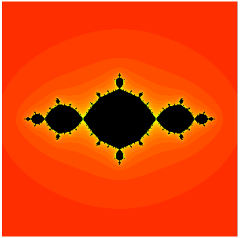
Black: $F^n(z)$ remains bounded. Colored: $F^n(z) \to \infty$ as $n \to \infty$.

Unit disk



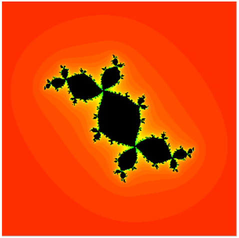
$c = 0$

Basilica



$c = -1$

Douady Rabbit



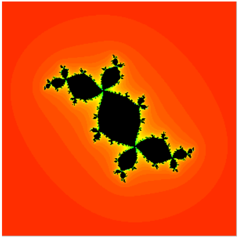
$c \approx -0.123 + 0.745i$

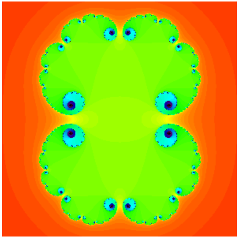
The Julia set is the boundary of the black region.

Preliminaries

Theorem (Fundamental Dichotomy)

The Julia set is either connected or completely disconnected (Cantor set).

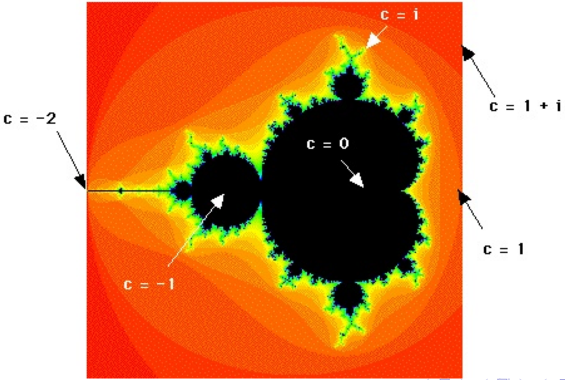




Mandelbrot Set

The **Mandelbrot set** $\mathcal{M}$ consists of all c values for which the filled Julia set of $J(z^2 + c)$ is connected. Equivalently,

$$\mathcal{M} = \{c \in \mathbb{C} : |F^n(0)| \nrightarrow \infty \text{ as } n \to \infty\}.$$



Classification of Periodic Orbits

Definition

The multiplier of an orbit with period n is given by

$$\Lambda = \frac{d}{dz} F^n(z_i) = F'(z_1)F'(z_2)F'(z_3)\dots F'(z_n).$$

The orbit is:

(1) $|\Lambda| < 1$ attracting

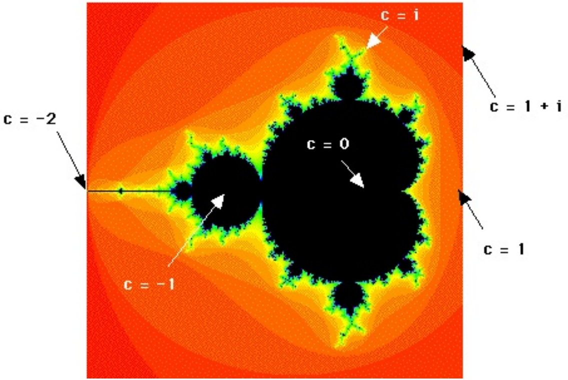
(2) $|\Lambda| > 1$ repelling

(3) $|\Lambda| = 1$ neutral

Hyperbolic components of the Mandelbrot Set

Definition

A map is **hyperbolic** if the orbit of every critical point converges to an attracting periodic orbit.



Julia sets of $z^2 + c$
The Basilica ($c = -1$), and the Douady Rabbit

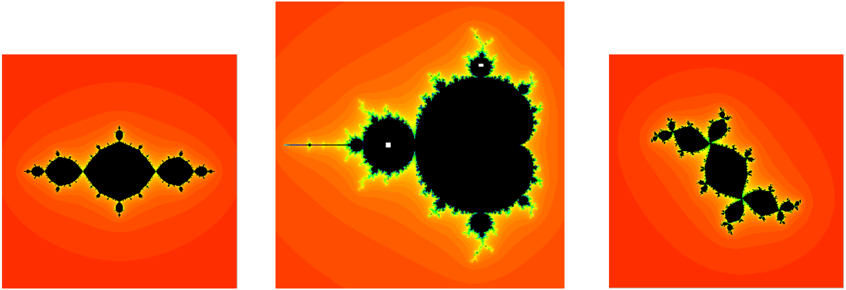


Figure : The Julia set called the basilica (left), the Mandelbrot set (center), The Julia set from the period 3 bulb (hyperbolic component) is called the Douady rabbit (left).

Understanding the Mandelbrot Set and Julia sets

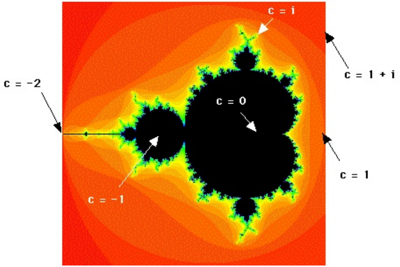
Mandelbrot set explorer animations

MLC Conjecture

Conjecture

The Mandelbrot set is Locally Connected (MLC).

Would imply we completely understood the dynamics of the simplest nonlinear function $z^2 + c$. Open of 30 years.



Introduction

Perturbed complex quadratic polynomial maps are functions of the form

$$F_\lambda(z) = z^2 + c + \frac{\lambda}{z^2} \quad \text{with } \lambda \in \mathbb{C}$$

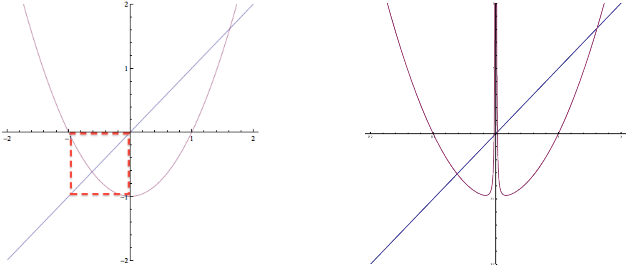


Figure : Illustration of unperturbed and perturbed quadratic maps with $c = -1$ and $\lambda \in \mathbb{R}^+$.

Main result of the talk

Theorem

Let c be the **center** *of the* **hyperbolic component** *of* **period** $n > 1$ *in the* **Mandelbrot set** *and let*

$$F_\lambda = z^2 + c + \frac{\lambda}{z^2}.$$

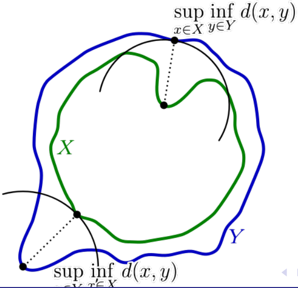
As $\lambda \to 0$, the filled **Julia sets** *of F_λ converge to the* **filled Julia set** *of the quadratic polynomial $F(z) = z^2 + c$ in the Hausdorff metric.*

Hausdorff metric

Definition

Let U and V be two non-empty subsets of a metric space (X, d). We define their Hausdorff distance $d_H(U, V)$ by

$$d_H(U, V) = \max\{\sup_{u \in U} \inf_{v \in V} d(u, v), \sup_{v \in V} \inf_{u \in U} d(u, v)\}$$

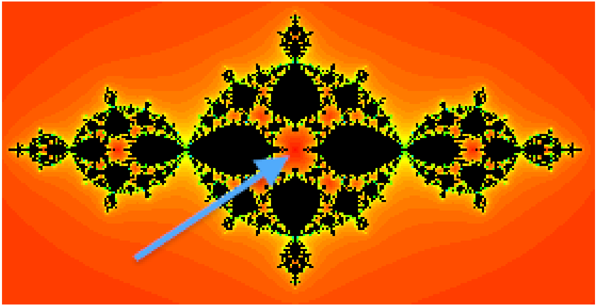


Trap Door

Definition

The region about the origin which is mapped to B_λ under one iteration is called the ***trap door*** *and is denoted T_λ.*

All colored regions within ∂B_λ are preimages of the T_λ.



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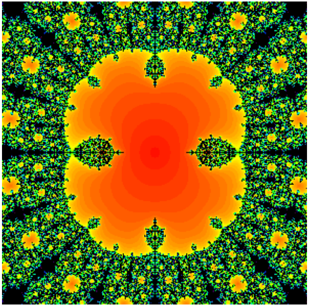
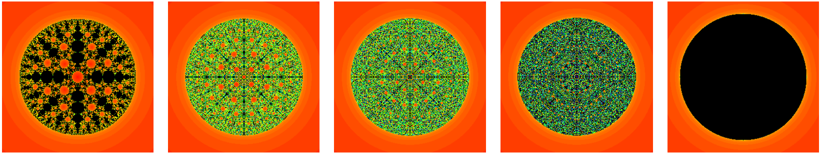
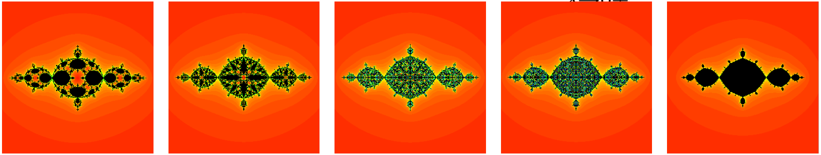


Perturbed Quadratic Map (PQM) $F_\lambda(z) = z^2 + c + \frac{\lambda}{z^2}$

Previous result '08: (Julia set of PQM with $c = 0$) $\xrightarrow[\lambda \to 0]{}$ unit disk



Basic Case: (Julia set of PQM with $c = -1$) $\xrightarrow[\lambda \to 0^+]{}$ filled basilica



$\lambda = 10^{-2}$ $\lambda = 10^{-3}$ $\lambda = 10^{-4}$ $\lambda = 10^{-6}$ $\lambda = 0$

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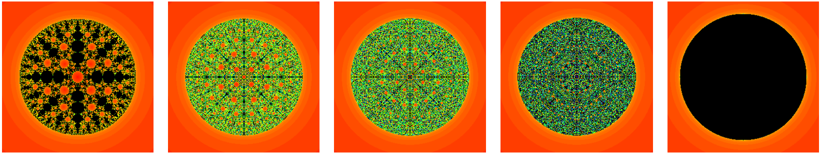
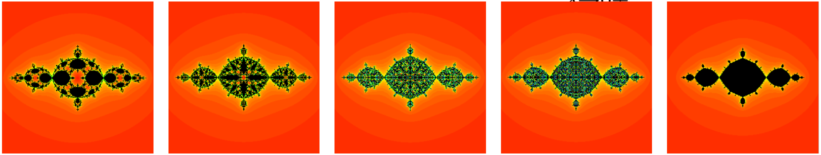


<u>Basic Case:</u> (Julia set of PQM with $c = -1$) $\xrightarrow[\lambda\to 0+]{}$ filled basilica

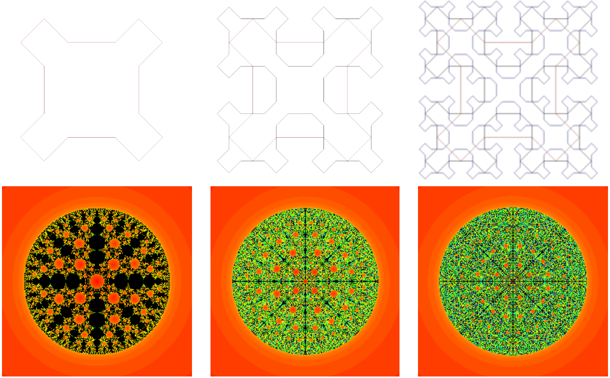


$\lambda = 10^{-2}$ $\lambda = 10^{-3}$ $\lambda = 10^{-4}$ $\lambda = 10^{-6}$ $\lambda = 0$

<u>Theorem:</u> Julia set of PQM with c in the center of any periodic bulb in the Mandelbrot set $\xrightarrow[\lambda\to 0]{}$ filled Julia set.

On Convergence

Space-filling curve (Sierpinski curve)



Significance

If a Julia set contains an open set, the the Julia set must be the entire Reimann Sphere.

We give Julia sets that come arbitrarily close to containing an open set.

Previous Work at BU: Result due to Blanchard et al.

The boundary of the immediate basin of attraction of ∞, B_λ, of a perturbed rational map is homeomorphic to that of the unperturbed map.

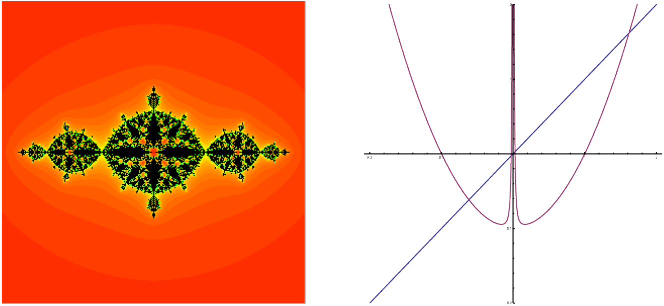


Figure : The outer boundary of $J(F_\lambda)$ is shaped like the basilica.

On Critical Points

The critical orbit determines the structure of the Julia set.

For rational maps:

(1) Critical points lie in $B_\lambda \Rightarrow$ Cantor set

(2) Critical points lie in $T_\lambda \Rightarrow$ Cantor set of curves

(3) All other cases $\Rightarrow$ Connected

Work by Mark Morabito et al.

Proves that for $G_\lambda(z) = z^n + \lambda/z$ and $n \geq 2$ the Julia set converges to the unit disk when $\lambda \to 0$ along $n-1$ special rays.

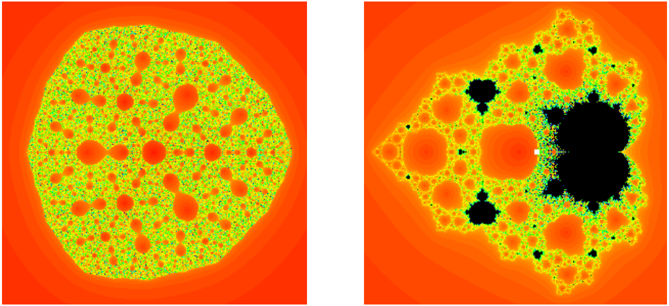


Figure : The Julia set (left), the Multibrot set (right).

Work by Antonio Garijo et al.

$H_\lambda(z) = z^n + \lambda/z^d$ in the case of $n, d > 2$ there always exists an annulus of fixed size in the complement of the Julia set for all λ ...

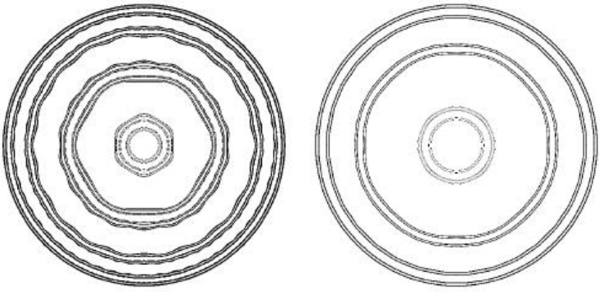


Figure reproduced from original paper.

Work by Antonio Garijo et al.

...but $H_\lambda(z) = z^n + \lambda/z^d$ converges to the unit disk in the case of $n = d = 2$ as $\lambda \to 0$.

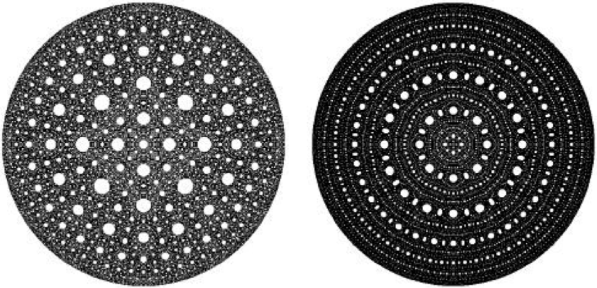


Figure reproduced from original paper.

$F_\lambda(z) = z^n + c + \frac{\lambda}{z^d} \quad n, d > 2$

Cubic Mouse

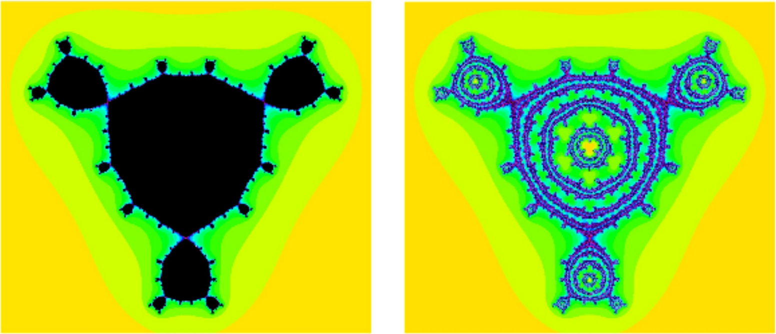
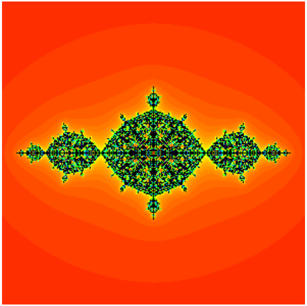


Figure 1: The Julia sets for $z^3 - i$ (left) and $z^3 + \lambda/z^3 - i$ when $\lambda = 0.0001$ (right). The critical point in the unperturbed map, $z^3 - i$, is in a superattracting 2-cycle. Note the explosion that occurs when λ becomes nonzero.

Part I: Julia Sets converge to the filled basilica for $F_\lambda = z^2 + c + \frac{\lambda}{z^2}$ and $\lambda \to 0+$

Julia set for $c = -1$



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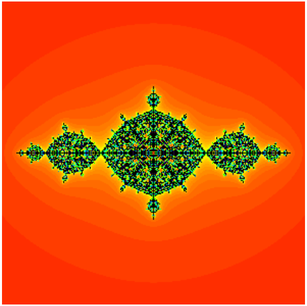


Part I: Julia Sets converge to the filled basilica for $F_\lambda = z^2 + c + \frac{\lambda}{z^2}$ and $\lambda \to 0+$

Steps of Proof:

1. None of the critical points escape for small $\lambda \in \mathbb{R}^+$
2. Existence of invariant interval in the Julia set
3. Symbolic Dynamics
4. Construction of a Cantor Necklace
5. Finish by showing that iterates of a disk of radius ϵ in the central bulb will intersect with the Cantor necklace in $J(F_\lambda(z))$.

The Critical Points

- The critical points determine the structure the Julia set.
- To find the critical point we need to solve the following equation for z_0 in terms of λ:

$$|F'_\lambda(z_0)| = (z^2 - 1 + \frac{\lambda}{z^2})' = 2z - \frac{2\lambda}{z^3} = 0 \tag{1}$$

$$z_0 = \lambda^{1/4} \tag{2}$$

- In $\mathbb{C}$ there are four distinct critical points.

Critical Values

The four complex critical points $\lambda^{1/4}$ are mapped 2-to-1 onto the two critical values on the real axis by F_λ.

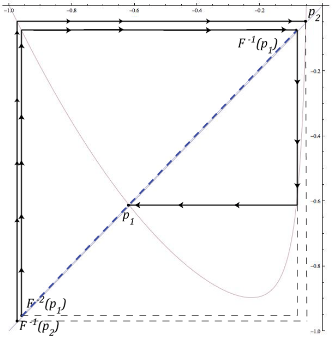
$$v_\lambda = -1 \pm 2\sqrt{\lambda}$$

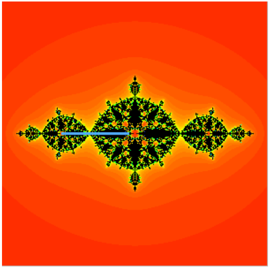
Show: Neither critical value escapes eventually.
The third iterate of the critical points are at $-15/16$, which is a constant distance from v_λ as $\lambda \to 0$.

The Invariant Interval/1

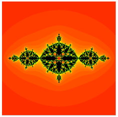
Proposition

There is an invariant interval I in the filled Julia set under the map F_λ^2 which connects T_λ and the preimage of T_λ in D_1.

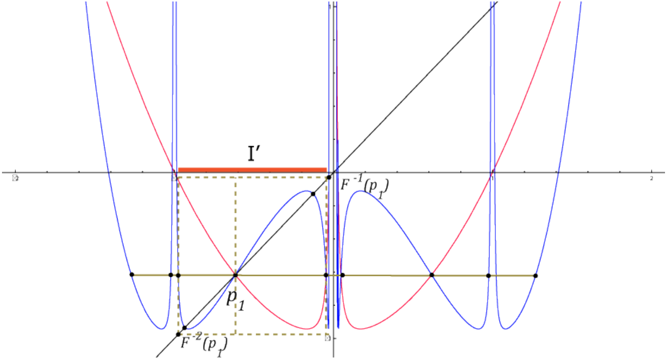




Graphical analysis of the first and second iterate of $F_\lambda(z)$.

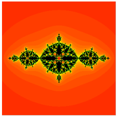


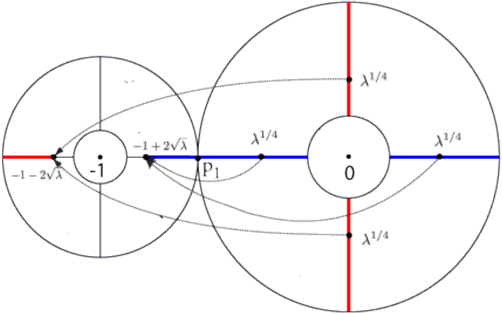
The Invariant Interval/2



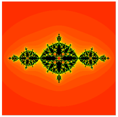
Graphical analysis of the first and second iterate of $F_\lambda(z)$. We show there is a invariant interval under the second iterate.

The Invariant Interval/3

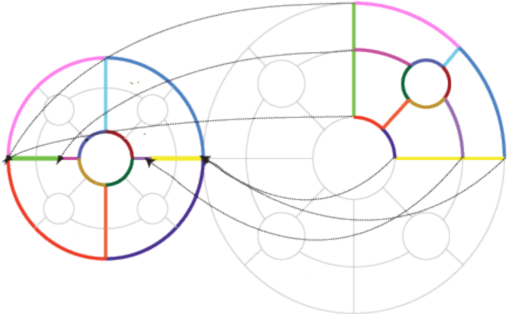




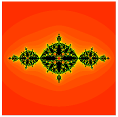
Schematic illustration of two regions in the basilica Julia set. Segments a given same color are mapped to the segments of the same color.



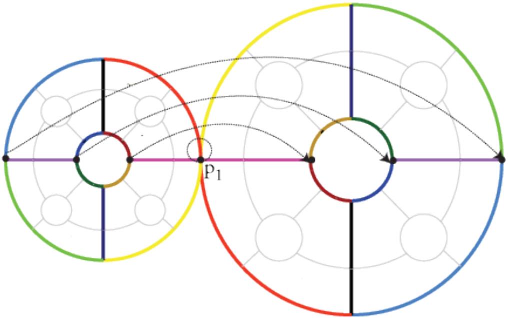
Behavior of the 1st Iterate of a Sector



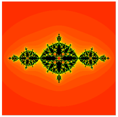
Segments given a same color are mapped to the segments of the same color.



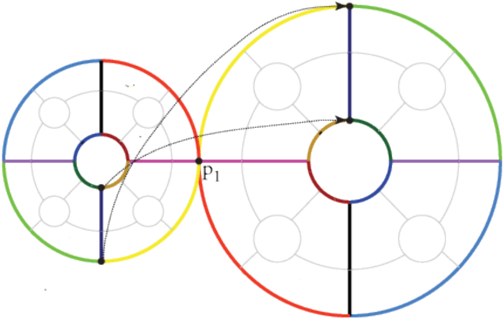
Behavior of the 2nd Iterate of a Sector/1



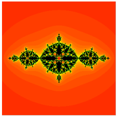
Segments given a same color are mapped to the segments of the same color.



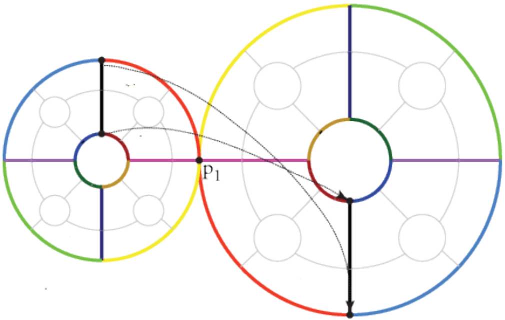
Behavior of the 2nd Iterate of a Sector/2



Segments given a same color are mapped to the segments of the same color.



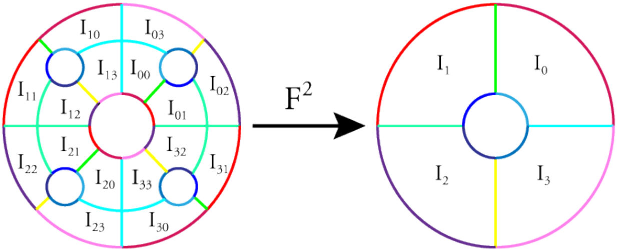
Behavior of the 2nd Iterate of a Sector/3



Segments given a same color are mapped to the segments of the same color.

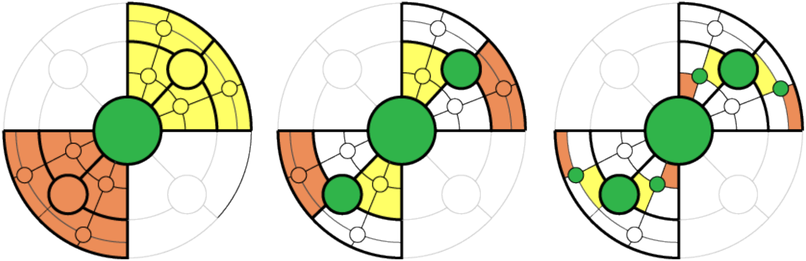
F_λ^2 maps the central bulb of the basilica onto itself

The mapping of central bulb D_0 under F_λ^2 is 4-to-1.



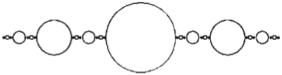
The 4 preimages of each of the fundamental sectors of the central bulb. Its four preimages in $J(F_\lambda)$ under F_λ^2 are shown. Segments of the same color are mapped to the segments of the same color.

1st, 2nd, . . . preimages of 2 central bulb sectors of J



Green: preimages of B_λ; Yellow and Brown sectors: trace the evolution of the preimages.

This is homeomorphic to the Cantor middle thirds necklace;



generalization of Devaney (2006).

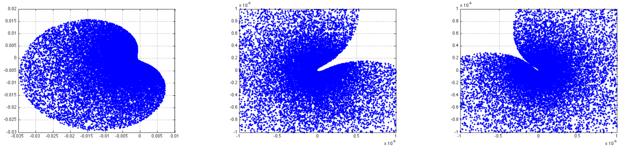
Final Step of the Proof

Theorem (Disk Wrapping around the origin)

Let $B_\epsilon(z_0)$ denote the disk of radius $\epsilon > 0$ centered at z_0. There exists a $\mu > 0$ such that, for any $\lambda \in \mathbb{R}^+$ such that $0 < |\lambda| \leq \mu$, $J(F_\lambda) \cap B_\epsilon(z_0) \neq \emptyset$ for all $z_0 \in D_0$.

- The $2k$-th iterates of disk always stays in central bulb D_0.
- $\exists n,\ F^{2n}(B_\epsilon)$ wraps around origin, intersects Cantor necklace.
- Julia set is backwards invariant.
- This concludes the proof of the main result. $\square$

Julia set of F_λ for $\lambda \to 0+$ converges to the filled basilica.



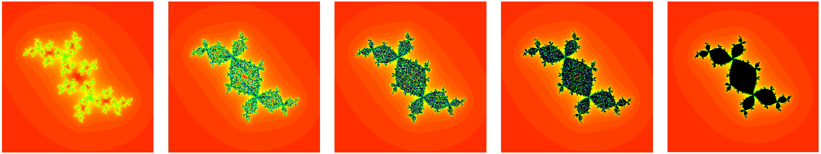
Part II: General case $F_\lambda = z^2 + c + \frac{\lambda}{z^2}$

Theorem

The Julia set of

$$F_\lambda = z^2 + c + \frac{\lambda}{z^2}$$

converges to the filled Julia set if c lies in the center of a hyperbolic component of the Mandelbrot set.



$\lambda = 10^{-2}$ $\lambda = 10^{-3}$ $\lambda = 10^{-4}$ $\lambda = 10^{-6}$ $\lambda = 0$

Steps of proof

1. Critical orbit remains bounded
2. Invariant circles in the Julia set
3. Rule out the existence of multiply connected Fatou components. (Herman rings, annular preimages of disks, etc.)
4. Fatou set within ∂B_λ does not contain a disk of radius ϵ

1. Behavior of the Critical Orbit

$$F_\lambda(z) = z^k + c + \frac{\lambda}{z^d}$$

Known that for F_λ with $k, d > 2$, and c the center of a hyperbolic component with period n, after $n+1$ iterations the critical points eventually escape to ∞. This means that all components of the Julia set contain a Cantor set of Simple closed curves.

Summary of Quadratic Cases

•Garijo: For main cardoid $z^2 + \frac{\lambda}{z^2}$ the 2nd iterate is $1/4$

•Previously: For period 2 bulb $z^2 - 1 + \frac{\lambda}{z^2}$ the 3rd iterate is $-15/16$

•Prove similar behavior for $z^2 + c + \frac{\lambda}{z^2}$ any c in the center of a period n bulb. Hence components of the Fatou set are simply connected. Show n–th iterates of the critical pints don't lie in the trap door.

Critical point is mapped "far away" from the critical value

Theorem

Let $F_\lambda(z) = z^2 + c + \frac{\lambda}{z^2}$, where c is the center of a period-n bulb of the Mandelbrot set. In this case the n-th iterate of the critical value behaves as follows:

$$\lim_{\lambda \to 0} F_\lambda^n(c \pm 2\sqrt{\lambda}) = c + \kappa,$$

where $\kappa = \left(4^n \left(\prod_{i=1}^{n-1} F^i(0)\right)^2\right)^{-1} \neq 0$ *is some constant.*

Proof /1

Recall:
Critical points: $\lambda^{1/4}$
Critical values: $v_\lambda = c \pm 2\sqrt{\lambda}$

$$
\begin{aligned}
F_\lambda(c \pm 2\sqrt{\lambda}) &= (c \pm 2\sqrt{\lambda})^2 + c + \frac{\lambda}{(c \pm 2\sqrt{\lambda})^2} \\
&= c^2 + c \pm 4c\sqrt{\lambda} + O(\lambda) \\
&= c^2 + c \pm (2c)(2\sqrt{\lambda}) + O(\lambda) \\
&\approx F(c) + F'(F(0))(\pm 2\sqrt{\lambda})
\end{aligned}
$$

Proof /2

$$F_\lambda^2(c \pm 2\sqrt{\lambda}) \approx F^2(c) + F'(F(0))F'(F^2(0))(\pm 2\sqrt{\lambda})$$

$$\vdots$$

$$F_\lambda^{n-1}(c \pm 2\sqrt{\lambda}) \approx F^{n-1}(c) + \prod_{i=1}^{n-1} F'(F^i(0))(\pm 2\sqrt{\lambda})$$

Know $F^{n-1}(c) = 0$ and $F'(z) = 2z$ hence

$$F_\lambda^{n-1}(c \pm 2\sqrt{\lambda}) \approx 2^{n-1}(\pm 2\sqrt{\lambda}) \prod_{i=1}^{n-1} F^i(0)$$

Critical orbit returns close to 0 after $n-1$ iterations.

Proof /3

The behaviour as $\lambda \to 0$:

$$F_\lambda^n(v_\lambda) \approx \left(2^{n-1}\left(\prod_{i=1}^{n-1} F^i(0)\right)(\pm 2\sqrt{\lambda})\right)^2 + c$$

$$+ \frac{\cancel{\lambda}}{\left(2^{n-1}\left(\prod_{i=1}^{n-1} F^i(0)\right)\cancel{(\pm 2\sqrt{\lambda})}\right)^2}$$

$$\approx c + \underbrace{\frac{1}{4^n\left(\prod_{i=1}^{n-1} F^i(0)\right)^2}}_{\kappa}$$

The n-th iterate of the critical orbit is not in T_λ

Theorem

If $|\lambda|$ is small enough, then $F_\lambda^n(\pm v_\lambda)$ both lie in D_1^λ.

Proof:

- $F^n|D_0$ is analytically conjugate to $z \mapsto z^2$ on $\mathbb{D}$
- Let $h : D_0 \to \mathbb{D}$ be the conjugacy with $h(0) = 0$
- Then $h(F^n(z)) = (h(z))^2$
- Suppose $h(z) = a_1 z + a_2 z^2 + \ldots$ so $h'(0) = a_1$

Proof /2

Find a_1:

- $h(F^n(z)) = (h(z))^2$
- Leading coefficient RHS: $a_1^2 z^2$
- Leading coefficient LHS: $\left(\prod_{i=1}^{n-1} F'(F^i(0))\right) a_1 z^2$
- Hence $h'(0) = a_1 = \prod_{i=1}^{n-1} F'(F^i(0))$

The for the Reimann map $h^{-1} : \mathbb{D} \to D_0$ we have

$$(h^{-1})'(0) = \left(\prod_{i=1}^{n-1} F'(F^i(0))\right)^{-1}$$

Proof /3

Construct a similar Reimann map $\Phi : \mathbb{D} \to D_1$

- $F^{-1} : D_{j+1} \to D_j$ is an analytic homeomorphism.
- $\Phi = F^{-(n-1)} \circ h^{-1} : \mathbb{D} \to D_1$ maps 0 to c

Then

$$\begin{aligned}
\Phi'(0) &= (h^{-1})'(0) \cdot (F^{-1})'(0) \cdot (F^{-1})'(0)(F^{n-1})(0) \dots (F^{-1})'(0)(F^2)(0) \\
&= \left(\prod_{i=1}^{n-1} F'(F^i(0)) \right)^{-2} \\
&= \frac{1}{4^{n-1}} \cdot \frac{1}{\left(\prod_{i=1}^{n-1}(F^i(0))\right)^{-2}} = 4\kappa
\end{aligned}$$

Proof /4

Theorem (Koebe 1/4 Thoeorem)

Suppose that the map $\eta \mapsto \psi(\eta) = a_1\eta + a_2\eta^2 + \dots$ carries the unit disk $\mathbb{D}$ diffeomorphically onto an open set $U \subset \mathbb{C}$. Then the distance r between the image of the origin and ∂U satisfies $\frac{1}{4}|a_1| \leq r \leq |a_1|$.

- $\Phi'(0) = a_1 = 4\kappa$ so by the Koebe 1/4 Thm D_1 must contain a disk of radius $\geq \kappa$.
- Inequality is strict (Remiann map is not the "classical Koebe map")
- D_1 contains a disc of radius strictly larger than κ at c $\Rightarrow$ $\boxed{c + \kappa \text{ is contained in } D_1^\lambda}$

This case is special

The degree of a map $z^d + c + \frac{\lambda}{z^m}$ is given by

$$deg = d + m.$$

In the case $d = m = 2$

$$4 = 2 + 2$$

So the Koebe $1/4$ theorem applies.
If $deg > 4$ then we have different behavior. Perhaps better proof.

2. Invariant Circles in the Julia Set

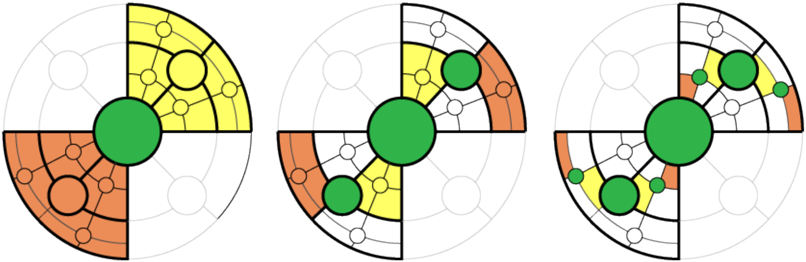
Analogue of the Cantor Necklace for general c.

- Let C_0 be the critical circle.
- H_λ be one of the involutions $\pm\frac{\sqrt{\lambda}}{z}$
- $C_{-1} = H_\lambda(F_\lambda^{-n}(C_0))$

Proposition

For $|\lambda|$ sufficiently small, there is a closed curve in the Julia set that is invariant under F_λ^n and that lies strictly between the curves C_0 and C_{-1}.

2. Invariant Curve in the Julia Set



Green: preimages of B_λ; Yellow and Brown sectors: trace the evolution of the preimages.

3. Fatou Components are simply connected

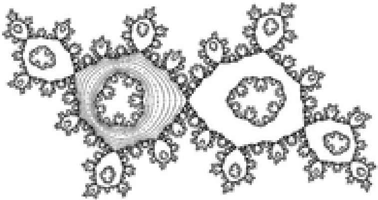
Classification of Fatou components

For every periodic component U of the Fatou set, exactly one of the following holds:

- U contains an attracting periodic point (attracting basin)
- U is parabolic
- U is a Siegel disc
  i.e. Conformally conjugate to irrational rotation of the disc
- U is a Herman ring
  i.e. conformally conjugate to irrational rotation of the annulus

3. Fatou Components are simply connected

Siegel disc



Herman ring

3. Fatou Components are simply connected

Theorem

If λ is small enough, then all Fatou components of F_λ are simply connected.

- If V is a simply connected Fatou domain of F_λ then all preimages of V are also simply connected.
- The Fatou set of F_λ never contains a Herman ring.
- There is no other type of periodic multiply connected Fatou component (basin of attracting or parabolic cycle).

3. Fatou Components are simply connected

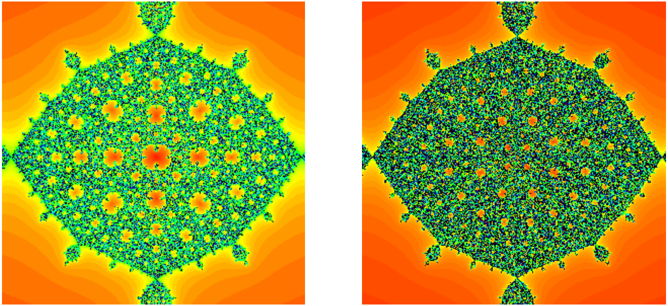


Figure : Central bulb D_0^λ for the basilica with $\lambda = 0.001$ and $\lambda = 0.0001$

Conclude Part II: Main Theorem Proof

Theorem

Let c be the center of the hyperbolic component of period $n > 1$ in the Mandelbrot set and let

$$F_\lambda = z^2 + c + \frac{\lambda}{z^2}$$

. As $\lambda \to 0$, the filled Julia sets of F_λ converge to the filled Julia set of the quadratic polynomial $F(z) = z^2 + c$ in the Hausdorff metric.

- Assume not
- The $k \cdot n$-th iterates of disk $B_\epsilon(z_*)$ in Fatou set stays in D_0^λ.
- $\exists l,\ F^{l \cdot n}(B_\epsilon)$ wraps around origin
- Ruled out multiply connected Fatou components
- Julia set is backwards invariant □

Movies!

References

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