Skálainvariáns dinamika táguló térben

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Introduction

Scale invariant trajectories – random walk, (fractional) Brownian motion

- $dX \sim dt^{\gamma}$ (eg. $\gamma = 1/2$)
  evetually slower than ballistic
- in 1D: all trajectories meet (with probability 1)

Our question:

- what if space expands faster than $\langle |dX| \rangle$?

Introduction – Expanding space

Cosmology

physicsforme.wordpress.com

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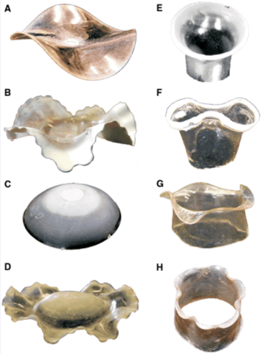
physicsforme.wordpress.com

Thin sheets



Klein, Efrati, Sharon, Science (2007)

Introduction – Expanding space

Cosmology

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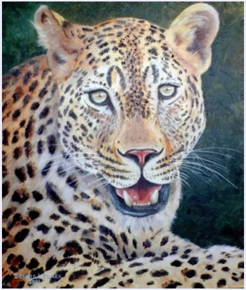
Growing substrate



stanford.edu/group/brainsinsilicon

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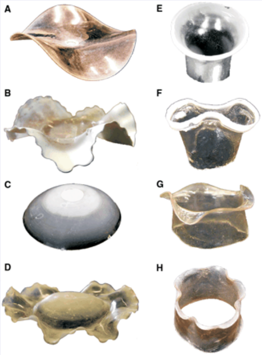

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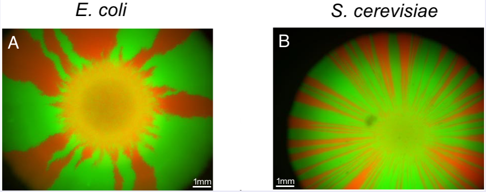
Indirect: domain walls



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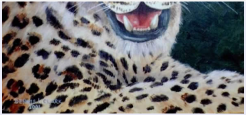

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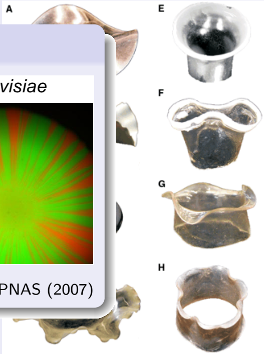
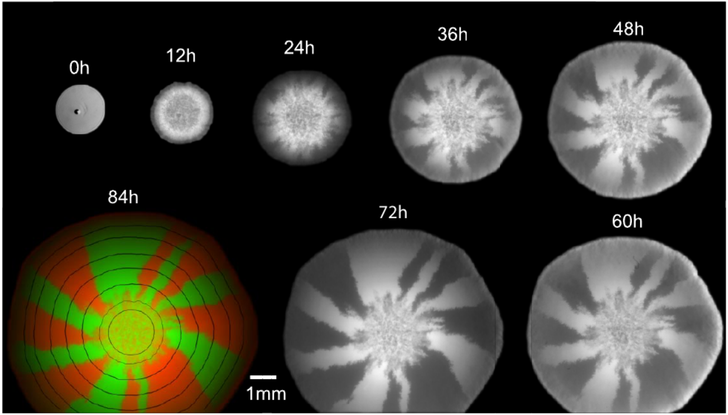


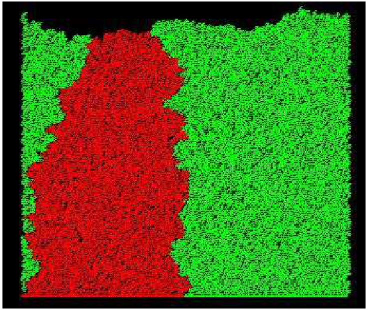
Klein, Efrati, Sharon, Science (2007)

Introduction – Genetic drift and range expansion



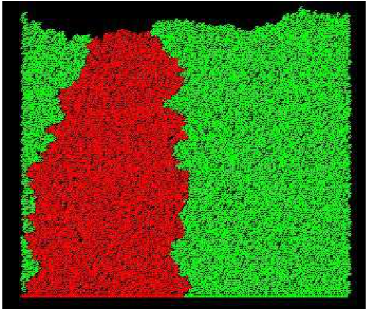
Hallatschek, Hersen, Ramanathan, Nelson; PNAS (2007)

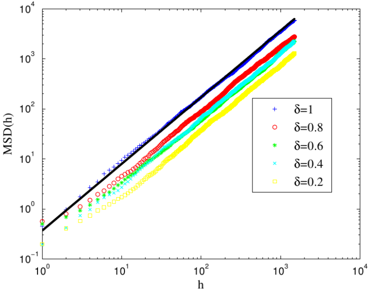
Domain boundaries



- domain boundaries grow perpendicular to surface
  X_h is superdiffusive due to surface roughness

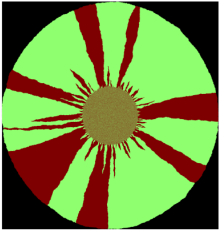
Domain boundaries

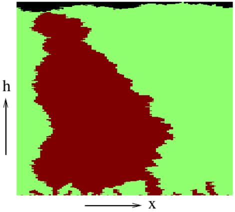




- domain boundaries grow perpendicular to surface
  X_h is superdiffusive due to surface roughness
- $M(h) := \langle X_h^2 \rangle \approx \sigma^2 h^{2\gamma}, \qquad \gamma = 2/3$

Range expansion

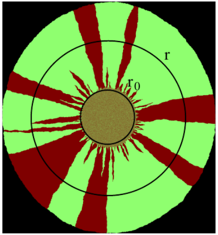


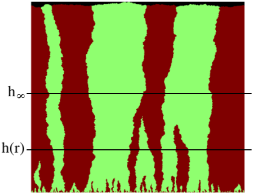


- fixed, finite size geometry: fixation (absorbing state in finite time)
- range expansion: promotes diversity and segregation

Mapping

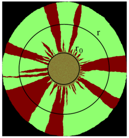
- understand radial growth
  with help of same dynamics in fixed size rectangular space

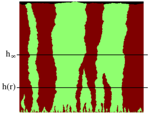




- polar-like coordinate transformation:
  linear: $0 \leq X_h < L, \quad 0 \leq h < \infty$
  radial: $0 \leq Y_r < 2\pi r, \quad r_0 \leq r < \infty, \quad 2\pi r_0 = L$

Mapping





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$$Y_r = \frac{r}{r_0} X_h$$

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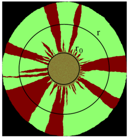

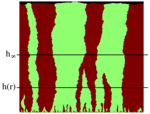

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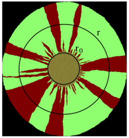
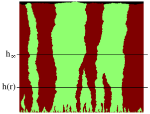
- increment:

$$dY_r = \frac{dr}{r_0} X_h + \frac{r}{r_0} dX_h = Y_r \frac{dr}{r} + d\tilde{Y}_r, \qquad \frac{r}{r_0} dX_h = d\tilde{Y}_r$$

Mapping





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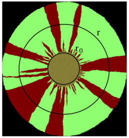
- preserving local structure:

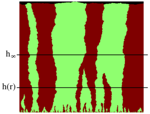
$$dX_h \sim (dh)^\gamma, \qquad d\tilde{Y}_r \sim (dr)^\gamma$$

$\gamma = 1/2$ diffusive fluctuations

$\gamma = 2/3$ KPZ domain boundaries

Mapping





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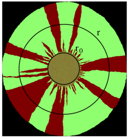
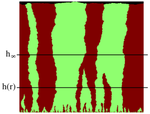
-

$$\frac{dh}{dr} = \left(\frac{dX_h}{d\tilde{Y}_r} \right)^{1/\gamma} = \left(\frac{r_0}{r} \right)^{1/\gamma}$$

Mapping





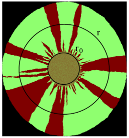
- integrating differential equation:

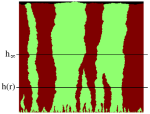
$$h(r) = \begin{cases} r_0 \frac{\gamma}{1-\gamma} \left(1 - (r_0/r)^{\frac{1-\gamma}{\gamma}}\right) & , \gamma \neq 1 \\ r_0 \ln(r/r_0) & , \gamma = 1 \end{cases}$$

- local interaction:

$$\left\{ \frac{r_0}{r} Y_r \right\} \stackrel{\text{dist.}}{=} \left\{ X_{h(r)} \right\} \quad \text{for all } r \geq r_0$$

Mapping





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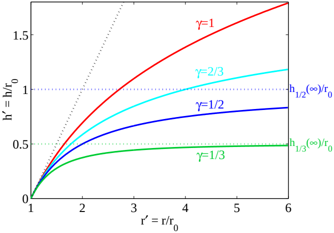
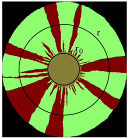


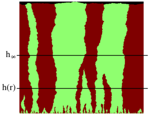
- properties:

  $h(r) \approx r - r_0$ for r close to r_0

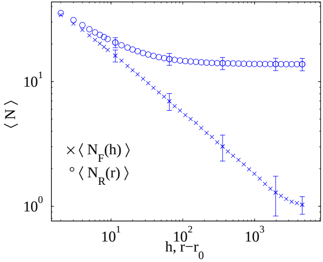
  $h_\gamma(\infty) = \lim_{r\to\infty} h(r) = \frac{\gamma}{1-\gamma} r_0 < \infty$

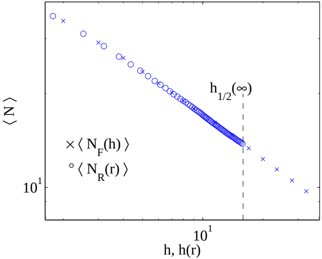
Results





- number of domains:





Results

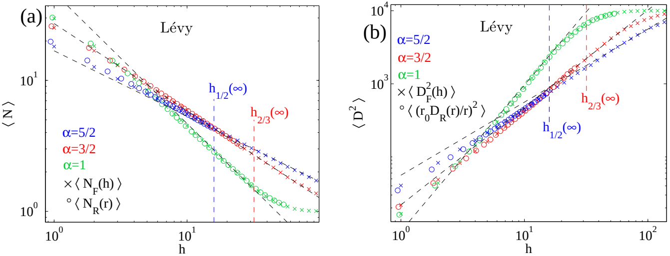
Different processes:

- Lévy flight:
  jump size distribution: $\mathbb{P}(X_{h+} - X_h = x) \sim C|x|^{-(1+\alpha)}$ $\quad (\alpha > 0)$
  Markovian
  $\gamma = \max\{1/\alpha, 1/2\}$

- fractional Brownian motion:
  covariances: $\langle X_{h+\Delta h} X_h \rangle \sim (h+\Delta h)^{2\gamma} + h^{2\gamma} - (\Delta h)^{2\gamma}$
  non-Markovian

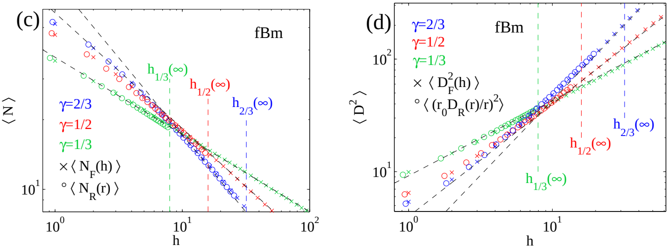
Results

Lévy flight ($\gamma = \max\{1/\alpha, 1/2\}$) + absorption:
number of surviving trajectories N,
mean square distance: $D_F(h)^2 = \sum_{i=1}^{N(h)} \left(X_h^{(i+1)} - X_h^{(i)} \right)^2$



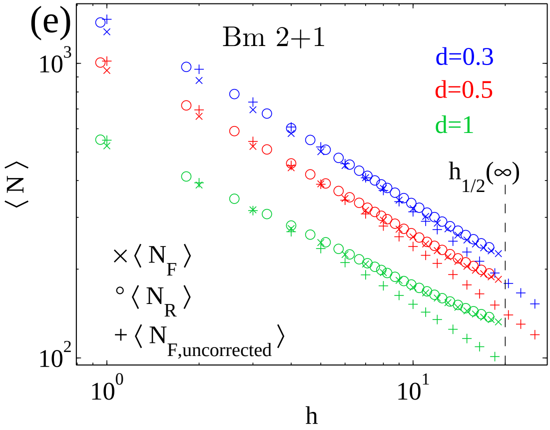
Results

fractional Brownian motion:



Results – finite particle size

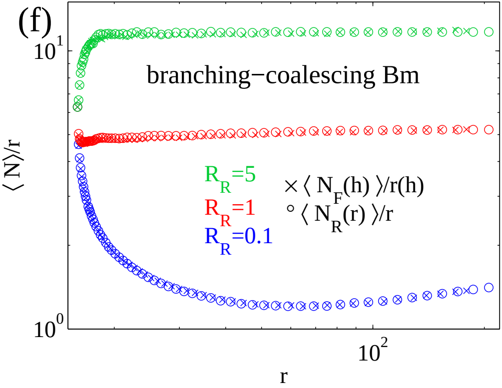
fixed size d in expanding space corresponds to
decreasing size $\frac{r_0}{r(h)} d$ in fixed space



Results – branching-coalescing Brownian motion

branching rate R_R in expanding space vs.
branching rate R_F in fixed space:

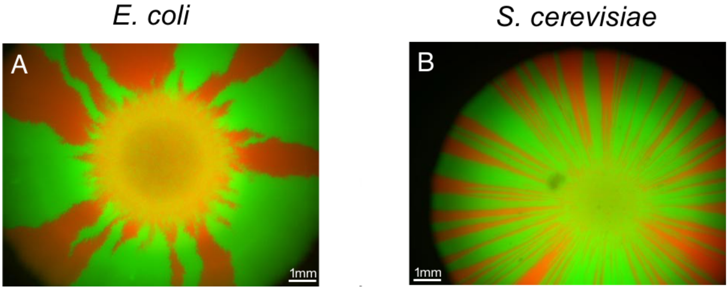
$$\frac{R_R}{R_F} = \frac{\Delta_R(dr)/dr}{\Delta_F(dh)/dh} = \frac{dh}{dr} = \left(\frac{r_0}{r}\right)^{1/\gamma}$$



Conclusions – mapping

- locally scale invariant growth processes
  mapped from homogenously expanding space to fixed space
- can be used eg. to handle 2D radial growth,
  asymptotic state in radial growth corresponds to finite time in fixed space
- can be extended to include:
  - finite particle size,
  - branching-coalescing,
  - higher dimensions etc.

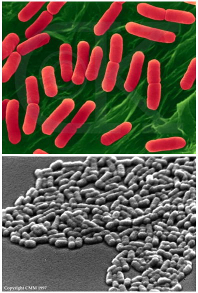
Bacteria vs. yeast



Hallatschek, Hersen, Ramanathan, Nelson; PNAS (2007)

Biology

Bacteria (E. Coli)



emc.maricopa.edu

Yiest (S. cerevisiae)

visualphotos.com

- ignore many biological details (shape, growth direction, etc.)
- consider: correlations due to reproduction time
- consider: overall geometry

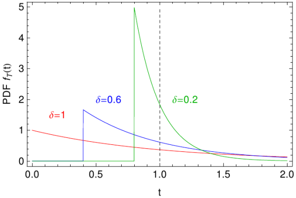
Model

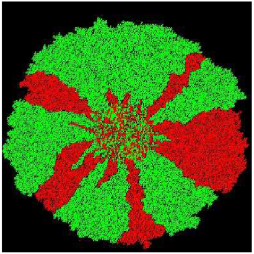
- **off-lattice Eden growth model**
- one-parameter family, $\delta \in [0,1]$: reproduction time T with delay $1-\delta$

  distribution: $T \sim 1-\delta+\mathrm{Exp}(1/\delta)$

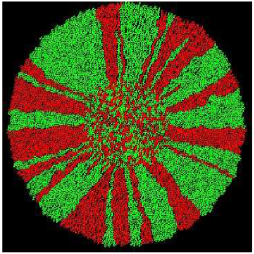
  normalized average: $\langle T \rangle = 1$

  variation coefficient: $\sigma(T)/\langle T \rangle = \delta$





$\delta = 1$



$\delta = 0.2$

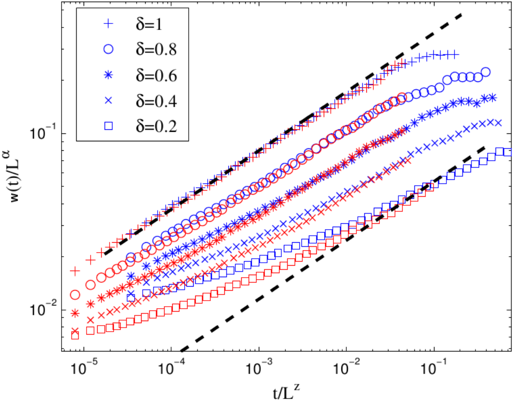
Surface – KPZ scaling

- on large scale:

$$\partial_t h(x,t) = v_0 + \nu \partial_x^2 h + \frac{\lambda}{2}(\partial_x h)^2 + D\eta(x,t)$$

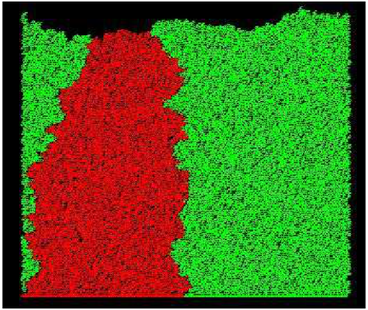
- scaling:
  $x \to x' = bx, \qquad t \to t' = b^{z} t, \qquad h \to h' = b^{\alpha} h$
  statistical scale invariance: $h(x,t) \sim h'(x',t')$
  Family-Vicsek scaling: $w(L,t) := \sqrt{\langle (h - \langle h \rangle_x)^2 \rangle_x} \sim L^{\alpha} f(t/L^z)$
  ahol $f(u) \sim \begin{cases} u^{\beta}, & \text{if } u \ll 1 \\ \text{const}, & \text{if } u \gg 1 \end{cases}$
- exponents: $\alpha + z = 2, \qquad z = \alpha/\beta$
  1D: $\alpha = 1/2, \quad \beta = 1/3, \quad z = 3/2.$

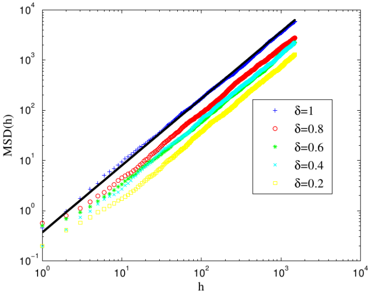
Surface – KPZ scaling



$$w(L,t) \sim \begin{cases} t^{1/3}, & \text{for} \quad t \ll L^{3/2} \\ L^{1/2}, & \text{for} \quad t \gg L^{3/2} \end{cases}$$

Domain boundaries





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- $M(h) := \langle X_h^2 \rangle \approx \sigma^2 h^{2\gamma}, \qquad \gamma = 2/3$

Correlations

Partial synchronization leads to intrinsic vertical correlations.

- $N(t)$ growth events with height Δh_i: $\quad h_{N(t)} = \sum_{i=1}^{N(t)} \Delta h_i$
- $\mathrm{var}[h_{N(t)}] = \langle \Delta h_i \rangle^2 \mathrm{var}[N(t)] + \langle N(t) \rangle \mathrm{var}[\Delta h_i]$

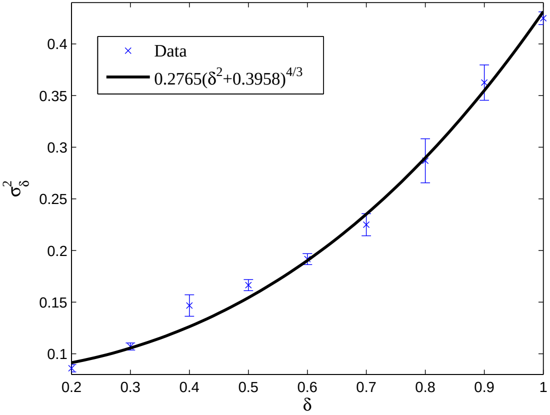
  $$= t \langle \Delta h_i \rangle^2 (\delta^2 + \epsilon^2) \stackrel{!}{=} O(1)$$

  where correlation coefficient due to geometric effects:
  $\epsilon = \sqrt{\mathrm{var}[\Delta h_i]} / \langle \Delta h_i \rangle$
- intrinsic vertical correlation: $\tau \sim \frac{1}{\delta^2 + \epsilon^2}$
  lateral correlation length: $\xi_{\parallel}(t) \sim (t/\tau)^{1/z}$
- mean square displacement
  $M(h) := \langle [X(h) - X(0)]^2 \rangle \approx \sigma_\delta^2 h^{2\gamma} \sim \xi_{\parallel}^2(h)$

$$\gamma = 2/3 \quad \text{and} \quad \sigma_\delta^2 \propto (\delta^2 + \epsilon^2)^{4/3}$$

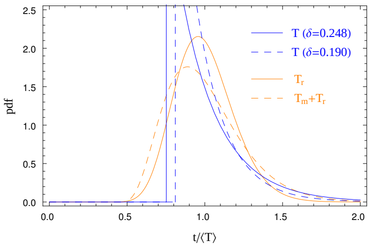
Correlations

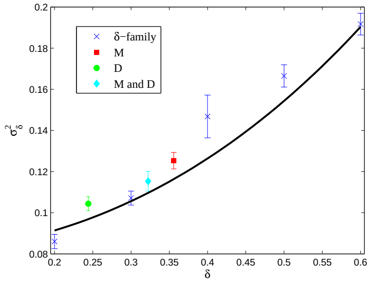
Prefactor of mean square displacement: $\sigma_\delta^2 \propto (\delta^2 + \epsilon^2)^{4/3}$



S. cerevisiae (yeast)

Replace δ-family with realistic reproduction times.





Conclusions – partially synchronized growth

- extension of off-lattice Eden model
  reproduction time has variation coefficient δ
- stays in KPZ universality class
- changes in patterns are due to changing prefactors – quantified
- works for realistic reproduction times (S. cerevisiae)