Scale Invariance, Critical Phenomena, and the Renormalization Group
Series 4

April 28, 2009

Exercise 4.1: Derive Griffiths' inequality between critical exponents

The free energy $F(T, M)$ is a concave function of T, that is

$$F(T_1, M) - F(T_2, M) \geqslant (T_1 - T_2)\frac{\partial F(T_1, M)}{\partial T_1} = -(T_1 - T_2)S(T_1, M) \tag{1}$$

(i) Specialize the above inequality for the case $(T_1, T_2, M) = (T, T_c, M)$, where $T < T_c$ and M is the equilibrium magnetization at T in zero field, to obtain

$$F(T_c, M) - F(T_c, 0) \leqslant F(T, M) - F(T_c, 0) - (T_c - T)S(T, M) \tag{2}$$

(ii) Note that, since $\partial F/\partial M = H$, at a $T < T_c$, with spontaneous magnetization M, one has the constant value $F(T, M_1) \equiv F(T, 0)$ for all M_1 satisfying $-M \leqslant M_1 \leqslant M$. Then use (1) at $(T_1, T_2, M) = (T_c, T, 0)$ to show that (2) leads to

$$F(T_c, M) - F(T_c, 0) \leqslant (T_c - T)(S(T_c, 0) - S(T, M)).$$

(iii) Assuming the following power laws

$$\frac{\partial F}{\partial M}(T_c, M) = H \sim M^{\delta}, \qquad M \sim (T_c - T)^{\beta}, \qquad C_H = T\frac{\partial S}{\partial T}\Big|_{H=0} \sim (T_c - T)^{-\alpha'},$$

where $T < T_c$, derive the inequality of Griffiths

$$\beta(\delta + 1) + \alpha' \geqslant 2,$$

based on the result in (ii).

Exercise 4.2: Show the correlation inequalities for the Ising chain.

Calculate the first two correlation functions of the ferromagnetic Ising chain in a homogeneous external field $G_1(T, h) = \langle s_0 \rangle = m(T, h)$ and $G_2(T, h, k) = \langle s_0 s_k \rangle$. The length of the chain goes to infinity in both directions while k can take any finite value. Show that the resulting functions satisfy the correlation inequalities. Hint: The correlation function G_2 is treated in the statistical physics textbook, for zero field it is given explicitly, now generalize the result for finite field.

Exercise 4.3: Derive Fisher's inequality $\gamma \leqslant \nu(2 - \eta)$.

While the proof can be found in the course book, here the task is to give an organized presentation of it in the class.