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Scale Invariance, Critical Phenomena, and the Renormalization Group
Series 1

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Exercise 1.1: Prove the Pythagorean theorem by dimensional analysis

Hints: The area A of a right triangle can be determined in terms of its say smallest angle ϕ and the length c of the hypotenuse. For reasons of dimensionality $A = c^2 \cdot f(\phi)$, where f is some function. Draw a right triangle, then discover two smaller ones inside precisely covering the area of the first one, and prove the theorem.

Exercise 1.2: Scaling in the Edwards-Wilkinson model of surface evolution

1.2a: For $1+1$ dimensions and spatially periodic boundary conditions, determine the time dependence of the average square roughness (width, thickness) of the surface. The square roughness of one instance of the surface $h(x,t)$ at time t is defined by

$$w_2(t) = \overline{\left(h(x,t) - \overline{h(x,t)}\right)^2}, \tag{1}$$

where overbar is spatial average $\overline{f(x)} = L^{-1}\int_0^L f(x)\,dx$. To be calculated is the ensemble average $\langle w_2(t)\rangle$ analytically in form of a series, then its small and large time asymptotes.

Hints: (i) Express $w_2(t)$ in terms of $c_n(t)$, the spatial Fourier coefficients of $h(x,t)$. Note that $c_n(t) = c^*_{-n}(t)$. (ii) Determine the Langevin equations for the $c_n(t)$'s, paying attention to the covariance of the noise. (iii) Solve the Langevin equations of (ii) explicitly in terms of the initial condition $c_n(0)$ and the noise. (iv) Substitute the $c_n(t)$'s into the formula found in (i), finally take the average over the noise. Thus an explicit sum formula is obtained for $\langle w_2(t)\rangle$. (v) Determine the large and small time asymptotes of $\langle w_2(t)\rangle$. In the latter case consider the initial state as flat (all $n \neq 0$ Fourier coefficients vanish), and calculate the asymptote of the sum via approximating it by an integral. (vi) Anyone is welcome to choose other paths of solutions.

1.2b: In the $1+d$ dimensional EW model determine the indices $(\alpha, z, \beta = \alpha/z)$ for which the change of variables $x' = bx$, $h' = b^\alpha h$, and $t' = b^z t$ leaves the equation of motion invariant. Based on these indices you can predict in $1+1$ dimensions, for small times, a pure power law for the average roughness as a function of time. Furthermore, determine the coefficient of the initial time dependence by dimensional analysis in terms of the parameters of the EW equation, then, for long times, give a formula for the saturation value of the average roughness again by dimensional analysis. Compare these results to the exact expression obtained in 1.2a.

Exercise 1.3: Show the absence of naive scale invariance, like the one in 1.2b, for the Kardar-Parisi-Zhang equation. Give a simple explanation from comparing the EW and KPZ equations why naive scaling is expected to break down for the latter. From simulations of the $1+1$ dimensional KPZ, asymptotes with $\alpha \approx 0.47$ and $\beta \approx 0.33$ can be extracted nonetheless; what opinion would you form based on this fact?